CONTROL THEORY FOR AIRCRAFT (R15A2113)

COURSE FILE

III B. Tech II Semester

(2017-2018)

Prepared By

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MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

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MRCET VISION

- To become a model institution in the fields of Engineering, Technology and Management.

- To have a perfect synchronization of the ideologies of MRCET with challenging demands of International Pioneering Organizations.

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To establish a pedestal for the integral innovation, team spirit, originality and competence in the students, expose them to face the global challenges and become pioneers of Indian vision of modern society.

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PROGRAM OUTCOMES

(PO's)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

3. **Design / development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.

6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a

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12. **Life- long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

EDUCATIONAL OBJECTIVES – Aeronautical Engineering

1. **PEO1 (PROFESSIONALISM & CITIZENSHIP):** To create and sustain a community of learning in which students acquire knowledge and learn to apply it professionally with due consideration for ethical, ecological and economic issues.

2. **PEO2 (TECHNICAL ACCOMPLISHMENTS):** To provide knowledge based services to satisfy the needs of society and the industry by providing hands on experience in various technologies in core field.

3. **PEO3 (INVENTION, INNOVATION AND CREATIVITY):** To make the students to design, experiment, analyze, and interpret in the core field with the help of other multi disciplinary concepts wherever applicable.

4. **PEO4 (PROFESSIONAL DEVELOPMENT):** To educate the students to disseminate research findings with good soft skills and become a successful entrepreneur.

5. **PEO5 (HUMAN RESOURCE DEVELOPMENT):** To graduate the students in building national capabilities in technology, education and research

PROGRAM SPECIFIC OUTCOMES – Aeronautical Engineering

1. To mold students to become a professional with all necessary skills, personality and sound knowledge in basic and advance technological areas.

2. To promote understanding of concepts and develop ability in design manufacture and maintenance of aircraft, aerospace vehicles and associated equipment and develop application capability of the concepts sciences to engineering design and processes.

3. Understanding the current scenario in the field of aeronautics and acquire ability to apply knowledge of engineering, science and mathematics to design and conduct experiments in the field of Aeronautical Engineering.

4. To develop leadership skills in our students necessary to shape the social, intellectual, business and technical worlds.

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III Year . Tech, ANE-II Sem	L	T/P/D	C
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(R15A2113) CONTROL THEORY FOR AIRCRAFT

Objectives:

- To acquire the student with method of modeling,
- Performance analysis of control system and
- Application to aircraft control system.

UNIT I: Control System modelling and feedback control:

Basic components of control system, open loop system, closed loop system, effect of feed back on overall gain, stability, sensitivity & on noise, Linear Vs Non- linear system, Time-invariant Vs time varying systems. Modelling of dynamical system by differential equations. Linearization of non-linear system. System type, steady state error, error constant. Composition, reduction of block diagrams of complex systems-rules and conventions. Control system components- sensors, transducers, servomotors, actuators, filters, modelling, transfer function.

UNIT-II: **Time Domain & Frequency Domain Analysis.**

Control system performance, time domain description, output response to control inputs. Characteristic parameters-relation to system parameters. Review of Laplace Transform, applications to differential equations, Poles and zeroes, partial fraction decomposition of transfer function. Frequency domain analysis, specification: resonant peak, resonant frequency and band width. Bode Plot, Polar plot. Experimental determination of transfer function by frequency response measurement.

UNIT-III: Design of Control System.

Control system performance requirements, transient and steady state specification. Example of first and second order system. Method of determining stability- Routh-Hurwitz Criterion. Design of controllers: active, passive, series, feed forward, feedback controller. Proportional, integral. Proportional plus derivative control. Lead, lag, lead-lag, wash-out, notch filters: properties and transfer functions. Gain scheduling, Adaptive control-definition, merits.

Stability of closed loop system, Root Locus method of analysis and compensation. Nyquist Criterion, gain margin and phase margin.

UNIT-IV : Aircraft response to control- Flying Qualities, Stability and Control Augmentation, Auto pilots.

Approximation to aircraft transfer functions, Flying qualities of aircraft, relation to airframe transfer function. Pilot opinion rating. Stability Augmentation system- displacement & rate feed- back, Full authority fly-by-wire control, need for automatic control. Auto pilots-purpose, functioning, displacement auto pilot, pitch, yaw, bank, altitude and velocity hold auto pilot. Auto pilot design by displacement feedback & series PID Controller- Zeigler and Nichols method.

UNIT-V: Modern Control Theory

Limitations of classical control system modelling, multi input multi output systems. State space modelling of dynamical systems, state variable-definition-state equations. The output variable-the output equation. Representation by vector matrix first order differential equations. Matrix transfer function, state transition matrix- matrix exponential, properties, Numerical solutions of state equations, examples. Canonical transformation of state equations, Eigen values, real distinct, repeated. Controllability and observability- definition-significance. Digital control system: over view- advantages, disadvantages.

Text Books:

1. KUO, BC. Automatic Control systems, prentice hall India, 1992 ISBN 0-87692-B3-0
2. Nelson R.C. Flight Stability and Automatic control, second edition, tata McGraw-hill2007 ISBN 0-07-666110-3
3. Yechout, T.R , Introduction to flight Mechanics, AIAA,2003,ISBN 1-56347-577-4

Reference: Mc Lean, D. Automatic flight Control Systems, prentice hall, 1990

Outcomes:

- The student should be able to model a control system.
- He should be able to estimate the performance of a specified control system including aircraft flight control system.
- He will have good understanding of modern control design methods.

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III B.TECH. II SEMESTER – AERONAUTICAL

CONTROL THEORY FOR AIRCRAFT (R15)

MODEL PAPER – I

MAXIMUM MARKS: 75

PART A **Marks: 25**

All questions in this section are compulsory

Answer in TWO to FOUR sentences.

a) what are the advantages/Disadvantages of open loop system compared to closed loop system?

b) Discuss the effect of feedback on overall gain.

c) Give the expression for the rise time of the step response for second order system.

d) Define transfer function.

e) Define steady state error constants.

f) Discuss merits of robust control.

g) Discuss need for automatic control.

h) Explain the purpose of auto pilot.

i) Discuss the limitation of classical control.

j) What is time invariant linear system

PART B **Marks: 50**

Answer only one question among the two questions in choice.

Each question answer (irrespective of the bits) carries 10M.

Section-I

2 a) Describe a SISO Single input single output) system and a MIMO (Multiple input and multiple output) system and explain how they are analyzed.

b) Describe non- linear system and discuss how they are linearized

OR

3a) Explain the role of feedback in stability augmentation, control augmentation and automatic control with example.

b) Discuss use of transducer, sensor and filter in control system.

Section-II

4a) Find the poles and zeros of a control system whose transfer function is given by

$G(s) = (s+3)/(s^2+7s+12$b) With example explain the significance of gain and phase margin

OR

5a) Discuss the significance of corner frequencies, resonant frequencies and peak gain of a second order system

b) Explain the experimental method of determining system transfer function by frequency response measurements.

Section-III

6a) Discuss the functioning of proportional plus derivative control.**?**

b) Discuss the Root Locus method.

OR

7(a) Discuss the purpose and functioning of lead, lag and wash-out filters.

b) Discuss Nyquist criterion.

Section-IV

8a) Discus the relationship between flying qualities and aircraft transfer function

b) Discuss Zeigler and Nicholas method.

OR

9a) Discuss the role of auto-pilot as stability augmenter.

b) Discuss briefly functioning of fly-by-wire control.

Section-V

10a) Define the state variable and state equations with examples.

b) Discuss the properties of state transition matrix

OR

11a) Discuss the significance of Canonical transformation of state equations

b) Discuss the advantages and disadvantages of digital control system.

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MODEL PAPER – II

MAXIMUM MARKS: 75

PART A **Max Marks: 25**

All questions in this section are compulsory

Answer in TWO to FOUR sentences.

1a) Define dynamical systems and list its components.

b) Define linear time invariant system.

c) Discuss the relationship between impulse response and transfer function.

d) What do you mean by frequency transfer function?

e) How are the steady state and transient response specified?

f) Discuss the problem with derivative control.

g) Explain the role of rate feedback in stability augmentation system.

h) Differentiate between reversible and irreversible control.

i) Define matrix transfer function.

j) Define controllability.

PART B **Max marks :75**

Answer only one question among the two questions in choice. Each question answer (irrespective of the bits) carries 10M.

Section-I

2. (a) For a unity feedback system given by G(s) = $\frac{20(s+2)}{s(s+3)(s+4)}$

Find the static error constants & find the steady error for r (t) = 3 u(t).

b) Explain about the standard test signals.

OR

3a) With example explain the method of modelling dynamical systems using differential equations.

b) Discuss modelling and transfer function of i) servomotor ii) actuators.

Section-II

4a) A control system is defined by the following differential equation. Find the output response y (t) using Laplace transform method. Assuming y(t) and dy(t)/dt are zero at t = 0.

$\frac{d2y(t)}{dt2}$ + 7 $\frac{dy(t)}{dt}$ +12 y (t) = u (t); where u t) is unit step unit.

b) Discuss Bode and Polar Plot

OR

5a) Discuss the significance of band width, resonant frequencies, peak gain in relation to second order system.

b) With example discuss the time domain specifications of second order control system.

Section-III

6a) Define and discuss the purpose of gain scheduling.

b) What are the methods of determining the stability of closed loop system

OR

7a) Discuss merits and constraints of non- linear control.

b) Discuss gain and phase margin with suitable examples.

Section-IV

8a) Discuss the flying qualities requirement of an aircraft. What is pilot's opinion rating?**)**

b) Discuss purpose and functioning of pitch, yaw and bank hold auto pilot.

OR

9a) Discuss the role of displacement and rate feedback in the design of stability augmentation system.

b) Discuss the role and purpose of displacement auto-pilot.

Section-V

10(a) What is observability? Explain the tests for observability.

b) Check whether the system represented by

$\begin{bmatrix} \dot{x}1 \\ \dot{x}2 \\ \dot{x}3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ u, is controllable or not.

OR

11. Write the advantages and disadvantages of digital control system over analogue control system.

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MODEL PAPER – III

MAXIMUM MARKS: 75

PART A **Max Marks: 25**

i. All questions in this section are compulsory

ii. Answer in TWO to FOUR sentences.

1a) Discuss the merits of open loop system.

b) Discuss the need for a stable system

c) Define and explain transfer function

d) What do you mean by polar plot

e) Define steady state error.

f) Describe the merits and demerits of non -linear system.

g) What do you mean by pilot's opinion rating?

h) Draw the block diagram of a pitch attitude hold auto-pilot.

i) Discuss the significance of canonical transformation.

j) What is matrix transfer function?

PART B **Marks: 50**

Answer only one question among the two questions in choice. Each question answer (irrespective of the bits) carries 10M.

Section-I

2a) Describe a SISO and MIMO system and explain how they are analyzed.

b) Discuss the importance of studying control system.

OR

3a) Discuss the purpose and functioning of various filters used in control systems.

b) How is overall system stability determined?

Section-II

4a) Discuss the second order system specifications in time domain.

b) Transfer function of a control system is s/((s+1)(s+2)). Find the response for the unit step input.

OR

5a) Write short notes on (i) Gain and phase shift . (ii) Resonant frequency.

b) Describe the relation between transfer function and impulse response.

Section-III

6a) What is compensator? Explain about lead compensator.

b) Discuss the merits and demerits of PID controller.

OR

7a) Write short notes on (i) Gain scheduling (ii) Adaptive control

b) Discuss phase margin and gain margin.

Section-IV

8a) discuss the steps to determine the transfer function of an aircraft.

b) Discuss Zeigler and Nichols method in design of controllers.

OR

9a) Write short notes on reversible and irreversible flight control system.

b) Differentiate between stability control system and control augmentation system.

Section-V

10a) Discuss the method of modelling dynamical system using state space equations.

b) Discuss general form of time invariant linear system.

OR

11a) What is controllability? How do you test the controllability of a system?

b) Discuss the advantages of digital control system over analogue control system

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MODEL PAPER – IV

MAXIMUM MARKS: 75

PART A **Max Marks: 25**

All questions in this section are compulsory. Answer in TWO to FOUR sentences.

1. (a) Define dynamical system.

(b) What do you understand by Time invariant linear system?

(c) Briefly discuss impulse and indicial response.

(d) What is the relation between transfer function and impulse response?

(e) Write the properties and application of wash-out filter.

(f) State Nyquist's criterion.

(g) Differentiate between reversible and irreversible control.

(h) Write the purpose of autopilots.

(i) Define state variable and state equation.

(j) Define observability.

PART- B **Maximum Marks: 50**

Answer only one question among the two questions in choice. Each question answer (irrespective of the bits) carries 10M.

Section-I

2. (a) Discuss deterministic and stochastic control system.

(b) Discuss application of feedback in stability augmentation system.

OR

3(a) Discuss merits of feedback control.

(b) Discuss modelling and transfer function of different filters used in aircraft control.

Section-II

4(a) Discuss frequency response method of control system design.

(b) Discuss Bode and Polar plots.

OR

5(a) Discuss the procedure of experimental determination of system transfer functions by frequency response measurements.

(b) Discuss the significance of resonant frequency and bandwidth.

Section-III

6(a) Discuss the application of proportional and integral control.

(b) Discuss implementation, application of adaptive control.

OR

7(a) Discuss the significance and interpretation of gain margin, phase margin.

(b) Discuss frequency response method of analysis and compensation in control system.

Section-IV

8(a) Discuss the response of an aircraft to pilot's control input and atmosphere.

(b) Discuss pole-zero and time-response specifications of flying quality requirements.

OR

9. (a) With help of block diagram explain the functioning and components of a displacement autopilot. (b) Discuss the functioning of normal acceleration command maneuvering autopilot.

Section-V

10.(a) Discuss state space modelling of dynamical system.

(b) Discuss the properties of state –transition matrix.

OR

11. Discuss the process of numerical solution of state equation.

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MODEL PAPER – V

MAXIMUM MARKS: 75

PART A **Max Marks: 25**

All questions in this section are compulsory. Answer in two to four sentences.

1. (a) Discuss sensitivity of output to control input in a feedback control system.

(b) What is the need for a robust control?

(c) Explain the difference between system parameters and characteristic parameters.

(d) What do you understand by gain margin and phase margin?

(e) Define steady state error.

(f) What do you mean by compensation through pole zero cancellation?

(g) What is the purpose of stability augmentation system?

(h) Bring out the purpose of feedback signals in autopilot.

(i) Differentiate between state variable and state equation.

(j) Define controllability.

PART- B **Maximum Marks: 50**

Answer only one question among the two questions in choice. Each question answer (irrespective of the bits) carries 10M.

Section-I

2(a) Discuss the procedure for analyzing SISO and MIMO system.

(b)Discuss linear and non-linear systems with examples.

OR

3. (a) Discuss the rules and conventions of reducing the block diagram of complex systems.

(b) Discuss the application of feedback control in control augmentation system and automatic systems.

Section-II

4. (a) Discuss the following:

(i) Poles and (ii) Dominant pole.

(b) Discuss the following: (i) Resonant frequency (ii) Peak Gain

OR

5(a) Discuss the purpose of Bode plot.

(b) Solve the following differential equation using Laplace transform.

$\frac{d2y(t)}{dt2}$ + 7 $\frac{dy(t)}{dt}$ +12 y (t) = u (t); where u t) is unit step unit. Assume y (t) and d y (t)/dt is 0 at t = 0.

Section-III

6(a) Discuss steady state and transient specifications of a second order system.

(b) Discuss following type of controllers: (i) Series controller (ii) Feedback controller

OR

7. Discuss frequency response method of determining the stability of a closed loop system.

Section-IV

8(a) Discuss how approximate aircraft transfer function is obtained.

(b) Discuss the role of rate feedback in stability augmentation system.

OR

9 (a) Discuss the purpose and functioning of fly-by-wire system.

(b) Discuss the need for automatic control.

Section-V

10. (a) discuss limitation of classical control theory when applied to MIMO systems.

(b) Explain the general form of linear time invariant system.

OR

11(a) Discuss the significance of Canonical transformation.

(b) Write the advantages and disadvantages of digital control systems.

UNIT-I

Control Systems modeling & feedback control

Index

1.1 Basic Components of control system. The basic components of a control system are:

(a) Input or Objective of control

(b) Plant or control system components

(c) Outputs or Results.

The basic relationship between these three components is shown in fig 1.1 below.

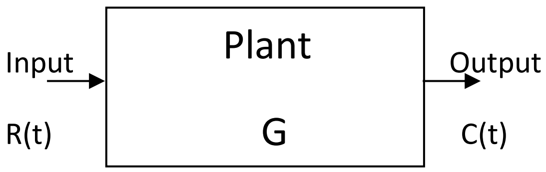


Fig: 1.1

In technical terms the objectives can be identified with inputs or actuating signal, u, and the results are called outputs or the controlled variable y. In general, the objective of a control is to control the output in some predetermined manner by the inputs through the elements of control systems.

Plant: a plant may be a piece of equipment, perhaps just a set of machine parts functioning together, the purpose of which is to perform a particular operation.

Process: any operation to be controlled is called a process. Examples are chemical, economic & biological processes.

Transfer Function. For any control system there exists an input termed as excitation or cause (denoted as R) which operates through a transfer operation termed as transfer function (denoted as G) and produces an effect resulting in output or response termed as controlled variable denoted as C). The cause and effect relationship between the output and input is related to each other through a transfer function. This relationship between the output and the input is represented by a diagram known as block diagram. The transfer function is expressed as the ratio of Laplace transform of output to Laplace transform of input with zero initial condition.

G(s) = Laplace transform of C(t)/ Laplace transform of R(t)

Block Diagram Representation. A control system may consist of a number of components. To show the function performed by each component, in control engineering, we commonly use a diagram called the block diagram. A block diagram of a system is pictorial representation of the function performed by each component and of the flow of signals. In block diagram all system variables are linked to each other through functional blocks. The functional block is a symbol for the mathematical operation on the input signal to the block that produces the output. The transfer function of the components is usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals. Fig1.2 below shows the elements of the block diagram.

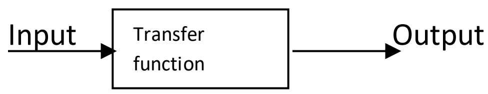


Fig 1.2

Block Diagram of a Closed Loop System for a single input single output System. Fig 1.3 below shows the block diagram of a closed loop system. The output C(s) is feedback to the summing point, where it is compared with reference input R(s). The output of the block C(s) is obtained by multiplying the transfer function G(s) by input to the block E(s). Output is measured by a sensor or measuring device whose transfer function is denoted by H(s).

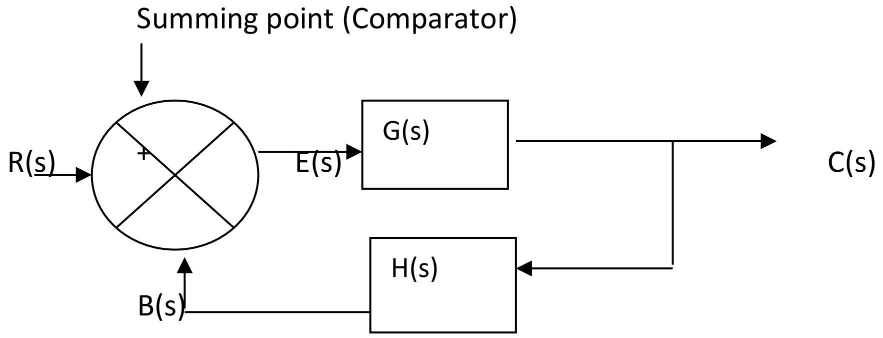


Fig 1.3

1.2 Open loop control, Closed loop Control, effect of feedback on overall gain, stability, sensitivity and on noise:1

1.2.1 The Concept of Feedback: a system that maintains a prescribed relationship between the output and the reference input by comparing them using the difference as means of control is called feedback control system.

1.2.2 Open Loop Systems: In open loop system, there is no feedback from output to input. Example of open loop control system is conventional washing machine, because amount of machine wash time is determined entirely by the judgment & estimation of the human operator. Another example is control of traffic light. The element of an open loop control system can be divided into two parts: the controller and the controlled process as shown below (fig 1.4)

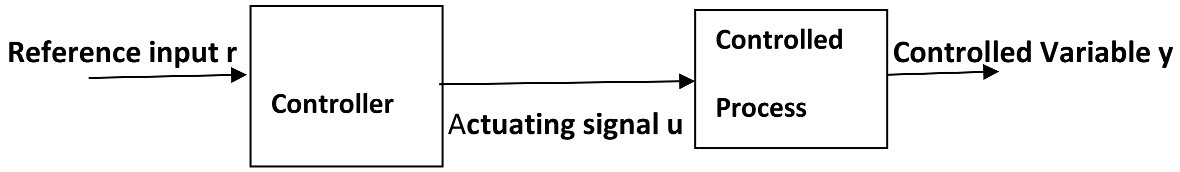


Fig 1.4: Elements of an Open-Loop system

An input signal or command r is applied to the controller, where output acts as the actuating signal u, the actuating signal then controls the controlled process that the controlled variable y will perform according to prescribed standards. In simple cases controller can be amplifier, mechanical linkage, filter or other control element, depending on the nature of the system. In more sophisticated cases, controller can be a computer such as a microprocessor. Open loop systems find application in many non-critical applications because of **simplicity and economy**.

1.2.3 Closed Loop Control System: What is missing in the open loop control system for more accurate & more adaptable control is the link or feedback from the output to input of the system. To obtain more accurate control, the controlled signal y should be fed back & compared with the reference input, and the actuating signal proportional to the difference of input and the output must be sent through the system to correct the error. Such a system is called closed loop system. Example of closed loop system is shown in the fig 1.4 below which is a room heating system.

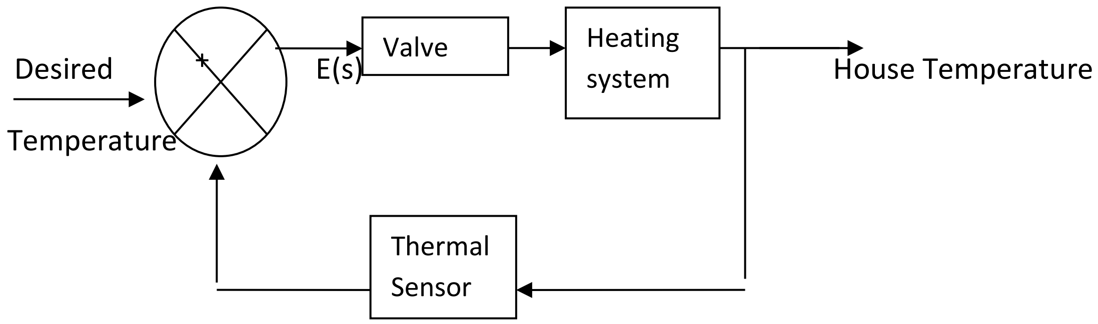


Fig 1.4: Home Heating System

A thermostat senses the temperature and if it is lower than a set value the furnace is turned on. The furnace is turned off when the temperature exceeds the set value.

Major Advantages of Open-loop control system are:

1. Simple construction and ease of maintenance.
2. Less expensive than corresponding closed loop system.
3. There is no stability problem.
4. Convenient when output is hard to measure or measuring the output precisely is economically not feasible. For example, in the washing machine, it would be quite expensive to provide a device to measure the quality of the washer's output, cleanliness of the clothes.

The major disadvantages of open loop systems are as follows:

1. Disturbance and changes in calibration cause errors, and the output may be different from what is desired.
2. To maintain the required quality in the output, recalibration is necessary from time to time.

1.2.4 Effect of feedback on overall gain, stability, sensitivity & on noise: In many control system application, the system designed must yield the performance that is robust i.e. insensitive to external disturbance, noise and parameter variations. Feedback in control system has the inherent ability of reducing the effect of external disturbance and parameter variations.

Feedback is not only for reducing the error between the reference input and the system output, it has many other significant effects on the performance of the control system. It has effects on such system performance characteristic such as stability, bandwidth, overall gain, disturbance and sensitivity.

Let us consider a simple example of feedback control system shown in fig 1.5

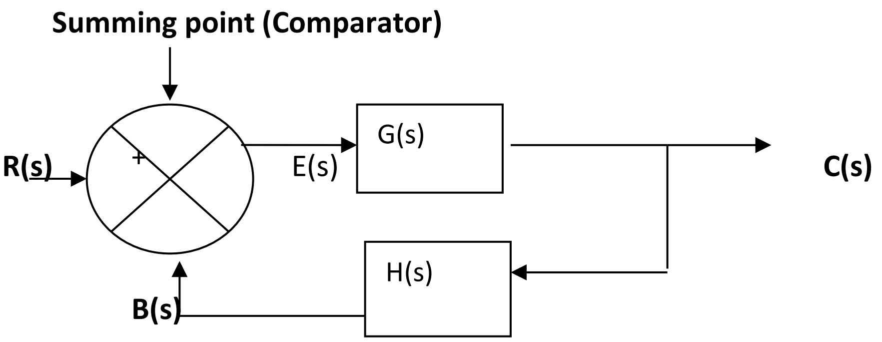


Fig 1.5

We know that overall gain of the system is

M(s) = Laplace transform of output/ Laplace transform of input.

$M(s) = C(s)/R(s) = G(s)/ (1 + G(s) H(s))$

(a) **Effect of feedback on overall gain**. The feedback affects the gain G(s) of a non-feedback system by a factor of 1+ G(s) H(s). The quantity G(s) H(s) may include a minus sign, so the general effect of a feedback is that it may increase or decrease the gain G(s). In practical control system, G(s) and H(s) are functions of frequency, so the magnitude of the 1+G(s) H(s) may be greater than one in one frequency range but less than one in other frequency range. So the feedback can increase the system gain in one frequency range but decrease it in other.

(b) **Effect of Feedback on Stability**. Stability is notion that describes whether the system will be able to follow the input command, or be useful in general. A system is said to be unstable if it is out of control. If G(s) H(s) = -1 the output is infinite for any finite input. Therefore, we may say that feedback can cause a system that is originally stable to become unstable. Feedback when used improperly can be harmful. It can be demonstrated that one of the advantages of incorporating feedback is that it can stabilize an unstable system.

(c) **Effect of Feedback on Sensitivity**. All physical elements have properties that change with the environment and age; we cannot always consider the parameters of a control system to be completely stationary over the entire operating life of the system. For example, winding resistance of an electric motor changes as the temperature of the motor rises during the operation. In general, a good control system should be very insensitive to parameter variations but sensitive to the input command. We consider G(s) to be gain parameter that may vary. The sensitivity of the gain of the overall system, M(s) to variation of G(s) is defined as

Sensitivity of M(s) with respect to G(s) =

$(\partial M(s)/M(s))/ (\partial G(s)/G(s))$ = % change in M(s)/ % change in G(s).

Sensitivity of M with respect to G = $1/ [G(s) + H(s)]$.

This relationship shows that if G(s) H(s) is positive constant, the magnitude of sensitivity can be made arbitrarily small by increasing G(s) H(s), provided the system remains stable. In open loop system sensitivity = 1. We should note that G(s) H(s) is a function of frequency, the magnitude 1 + G(s) H(s) may be less than unity over some frequency range, so that feedback could be harmful to the sensitivity to parameter variations in certain cases.

(d) **Effect of Feedback on External Disturbance or Noise**. The system with noise input n is shown in the figure 1.6

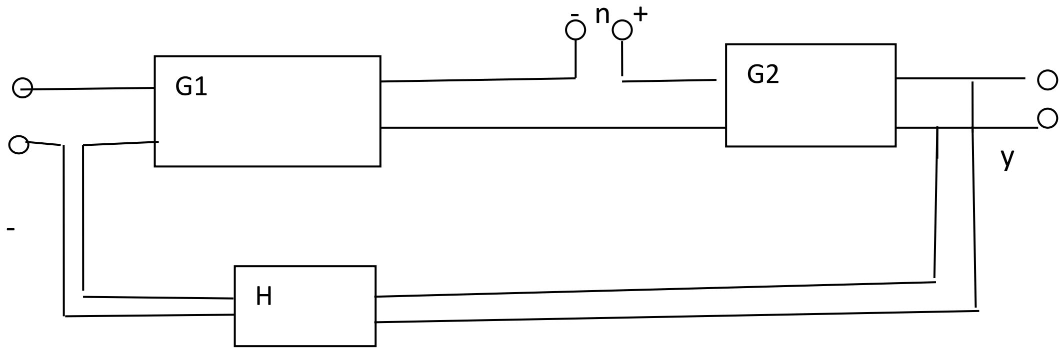


Fig 1.6: Feedback with external noise.

In the absence of feedback

Y= G2 n (1)

With presence of feedback, the system output due to noise n acting alone (i.e. r= 0)

Y = G2 n/ (1+ G1 G2 H) (2)

Comparing equation (1) and (2) shows that noise component in output is reduced by a factor of 1 + G1G2H. If the latter is greater than unity and system is kept stable.

In summary we can say that feedback if used properly will make the system robust by reducing the effect parameter variations, noise and external disturbance.

1.6 Linear and Non- Linear System, Time varying and Time invariant linear system:

Linear and Non- Linear System: A system is called linear if the principle of superposition applies. The principle of superposition states that the response produced by the simultaneous application of the different forcing functions is the sum of two individual responses. Hence for the linear system, the response to several inputs can be calculated by treating one input at a time and adding the results. It is this principle that allows one to build up complicated solutions to the linear differential equations from simple solutions. In an experimental investigation of a dynamical system, if cause and effect are proportional, thus implying that the principle superposition holds, then the system can be considered linear.

Example: An amplifier can be considered as linear system if output varies proportional to an input. This may be true if the input signal is not very large and amplifier does not saturate as shown in fig 1.7

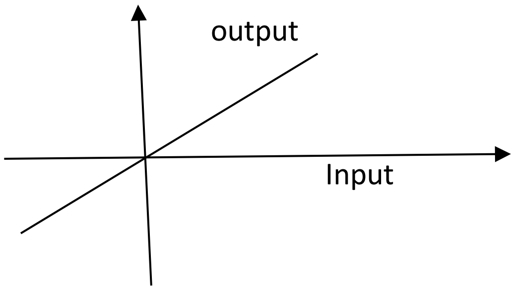


Fig 1.7: Linear system

A non- linear system is one where principle of superposition cannot be applied. Thus for nonlinear system the response to inputs cannot be calculated by treating one input at a time and adding the result. Although many physical systems are often represented by linear equations, in most cases actual relationships are not quite linear. In careful study of physical systems reveals that even so called linear systems are readily linear only in the limited operating range. For example, output of an amplifier may saturate for large input signals. There may be a dead space that affects the small signal. Dampers used in physical systems may be linear for low velocity operation but may become non- linear at high velocities, and the damping force may become proportional to the square of the operating velocity. This is shown in fig 1.8.

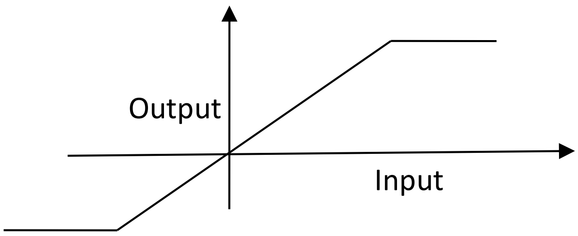


Fig 1.8: Non- Linear system

Time- Invariant System Linear System: When parameters of a linear control system are stationary with respect to time during the operation of the system, the system is called time invariant linear system. For example, in mass, spring damper system discussed above system parameters are spring constant k, damping force constant b and mass m. In case these parameters remain constant we say it is linear time invariant system. Most physical systems contain elements that drift or vary with temperature. For example winding resistance of the motor will vary when motor is first excited & its temperature is rising. In guided missile system, mass of the missile decreases as the fuel on board is being consumed during the flight.

1.7 **Modeling of Dynamical system by Differential Equation**:

A dynamical system can be modeled using the differential equations. The differential is derived by finding the relation between input and output using mathematical equations governing the system. This can be demonstrated using a mechanical system consisting of spring, mass, damper system as shown in fig 1.9.

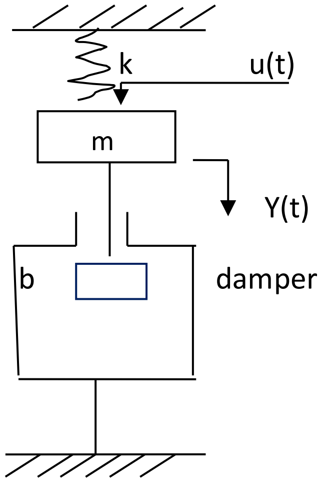


Fig 1.9: Mass, spring, damper system.

K is spring constant and b is coefficient of viscous damping. Spring force = k × x and Viscous force exerted by damper is b× dy/dt; where dy/dt is the velocity of the mass m. The external force u (t) is the input to the system and displacement y(t) is measured from the equilibrium position in the absence of the external force. The system is single input and single output system. We can write the system equation after drawing the free body diagram of the mass which is shown in fig: 1.10

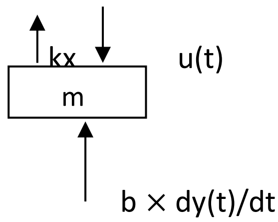


Fig 1.10: Free body diagram

From the diagram, the system equation is (using Newton's second law of motion):

$u(t) - k\,y(t) - b(\,dy/dt) = m\,d^2y/dt^2$

$u(t) = m\,d^2y/dt^2 + k\,y(t) + b\,dy/dt$

Taking the Laplace transform of both side

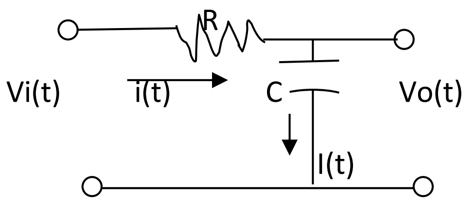
$u(s) = (\,ms^2 + b\,s + k\,)\,y(s)$

$y(s)/u(s) = G(s) = 1/(ms^2 +b s +k)$

Order of the system: It is order of the differential equation governing the input and output. In this case system is governed by second order differential equation; hence order of the system is two. Order of the system can also be defined as highest power of s in the denominator of the transfer function. In this example highest power of s is two; hence it is a second order system.

System Parameters: In the above example system parameters are mass m, spring constant k, and coefficient of viscous force b.

Another example of modeling dynamical system using differential equation:



In the above RC network which is also called low pass filter, input is applied voltage Vi(t) and Output is Vo(t). Resistance is R and capacitance is C. Let us derive the model of this system using differential equation.

Current passing through the circuit is i(t).

$Vi(t) = R\, i(t) + Vo(t)$

Charge on the capacitor $q = C\, Vo(t)$

Current i(t) is rate of change of charge q. hence

$dq /dt = c\, d\, Vo(t)/dt = i(t)$. Hence

$Vi (t) = RC\, d\, Vo(t)/dt + Vo(t)$

This is the first order differential equation; hence it is **first order system**. Solution of this equation will give the output for a given input. In this case **system parameters** are R and C.

Taking the Laplace transform of both sides and assuming zero initial condition we get

$Vi(s) = RCs\, Vo(s) + Vo(s)$

Therefore, transfer function $G(s) = 1/ (1 +RC\, s)$.

1.8 Linearization of Non- Linear System: To obtain a linear model we assume that variables deviate only slightly from some operating condition. Consider a system whose input is x (t) and output is y (t). The relationship between y(t) and x(t) is given by:

Y = f(x)

If the normal operating condition corresponds to $\bar{x}$, $\bar{y}$, then above equation may be may be expanded into a Taylor series about this point as follows:

$$y = f(x) = f(\bar{x}) + \frac{df}{dx}(x - \bar{x}) + \frac{1}{!2}\frac{d2f}{dx2}(x - \bar{x})^2 + \ldots$$

Where $\frac{df}{dx}$ and d^2f/dx^2 are evaluated at x = $\bar{x}$. If variation x - $\bar{x}$ is x = $\bar{x}$ small enough, we may neglect higher order terms in x - $\bar{x}$, then equation can be written as:

$$y = \bar{y} + k(x - \bar{x}); \quad (3)$$

Where $\bar{y}$ = f ($\bar{x}$) and k= $\frac{df}{dx}$ evaluated at x = $\bar{x}$

Then equation (3) can be written as:

$$y - \bar{y} = k(x - \bar{x}) \qquad (4)$$

Which indicates that y- $\bar{y}$ is proportional to x - $\bar{x}$. Equation (4) above gives linear mathematical model for the system given by equation (1) near the operating point x - $\bar{x}$, y - $\bar{y}$.

Next consider a non- linear system whose output y is function of two inputs x1 & x2, so that

$$Y = f(x1, x2) \qquad (5)$$

To obtain a linear approximation to this model, we may expand equation into Taylor series about point $\overline{x1}$, $\overline{x2}$. Then equation (5) is:

$$y = f(\overline{x1}, \overline{x2}) + \left[\frac{\partial f}{\partial x1}(x1 - \overline{x1}) + \frac{\partial f}{\partial x2}(x2 - \overline{x2})\right] + \frac{1}{!2}\left[\frac{\partial 2f}{(\partial x1)2}(x1 - \overline{x1})^2 + 2\frac{\partial 2f}{\partial x1 \partial x2}(x1 - \overline{x1})(x2 - \overline{x2}) + \frac{\partial 2f}{(\partial x2)2}(x2 - \overline{x2})^2\right] + \ldots$$

Where partial derivatives are evaluated at x = $\overline{x1}$, x2 = $\overline{x2}$. Near the normal operating point, the higher order terms may be neglected. The linear mathematical model of the non linear system in the neighborhood of the normal operating condition is given by

$$Y - \bar{y} = K1(x1 - \overline{x1}) + K2(x2 - \overline{x2})$$

Where $\bar{y}$ = f ($\bar{x}$, $\bar{y}$)

K1 = $\frac{\partial f}{\partial x1}$, evaluated at x1 = $\overline{x1}$, x2 = $\overline{x2}$

K2= $\frac{\partial f}{\partial x2}$, evaluated at x1 = $\overline{x1}$, x2 = $\overline{x2}$

Example problem: Linearize the nonlinear equation z = xy, in the region 5 ≤ x ≤ 7, 10 ≤ y ≤ 12. Find the error if the linearized equation is used to calculate the value of z when x=5; y=10.

Solution: Since the region considered is, 5 ≤ x ≤ 7, 10 ≤ y ≤ 12, choose $\bar{x}$ = 6, $\bar{y}$ = 1 then $\bar{z}$ = $\bar{x}$ $\bar{y}$ =66. Let us obtain a linearized equation for the nonlinear equation near a point $\bar{x}$ = 6 and $\bar{y}$ = 11.

Expanding the non- linear equation into Taylor series about the point x= $\bar{x}$,y= $\bar{y}$ and neglecting the high order terms,

z- $\bar{z}$ = K1(x-$\bar{x}$) + K2 (y-$\bar{y}$)

Where K1 = $\frac{\partial z}{\partial x}$ evaluated at x =$\bar{x}$, y = $\bar{y}$, K1 = 11

Where K2 = $\frac{\partial z}{\partial y}$ evaluated at x =$\bar{x}$, y = $\bar{y}$, K2= 6

Hence linearized equation is

z -66 = 11(x-6)+6(y-11) or z= 11x + 6y -66

When x=5, y=10, z= 11*5 + 6*10 -66 = 49.

The exact value is z=xy=50. The error is then 50-49=1. In percentage, the error is 2%.

1.9 System Type, Steady State Error, Error Constant:

System Type: A control system transfer function can be represented as:

G(s) = (K $(1+T_1s)(1+T_2s)\ldots(1+Tm_1\, s+Tm_2\, s^2))/(s^j\,(1+Ta\, s)(1+Tb\, s)\ldots(1+Tn1\, s+Tn2\, s^2))$

; Where K and all T's are real constants. The system type represents order of the pole of G(s) at s=0. Thus, the closed loop system having the forward path transfer function of above equation is type j, where j = 0, 1, 2… The following example illustrates the system type with reference to the form of G(s):

G(s) = K (1+0.5 s)/ ((s (1+ s) (1+2 s) $(1+s+s^2)$) **Type 1**

G(s) = K (1+ 2 s)/ (s^3) **Type 3**

Steady State Error: One of the objectives of control system is that the system output response follows a specific reference signal accurately in the steady state. The difference between the output and & reference input in the steady state is defined as the steady state error. In the real world, because of friction and other imperfection, & the natural composition of the system, the steady state output response seldom agrees with the reference. Therefore, steady state errors in control system are almost unavoidable. In a design problem, one of the objectives is to keep the steady state error to minimum, or below a certain tolerable value, and the same time the transient response must satisfy a certain set of specifications. The accuracy requirement on control systems depends to a great extent on the control objectives of the system.

Definition of steady state error with respect to system configuration: Let us consider a system as shown in fig 1.11 below. Error of the system may be defined as:

e (t) = reference signal – y(t)

Where reference signal in the signal that the output y(t) is to track. When the system has unity feedback (H (s) = 1), the input r(t) is the reference signal, and the error is simply

e (t) = r (t) – y(t)

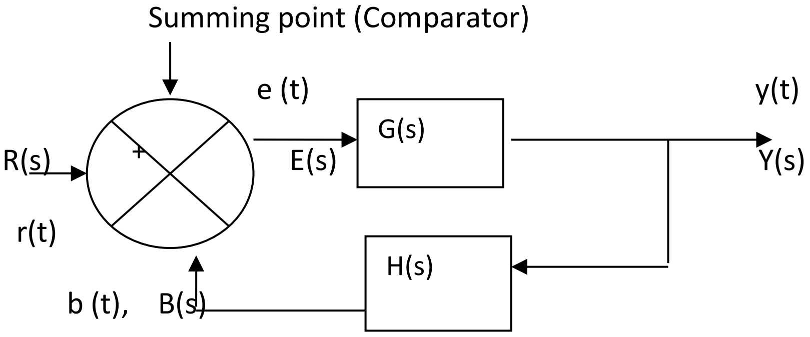


Fig 1.11. Feedback control system

The steady state error is defined as

$$\text{ess} = \lim_{t\to\infty} e(t)$$

Types of Control System: Unity feedback system. Consider a control system with unity feedback. It can be represented by or simplified to the block diagram in fig 1.12 below.

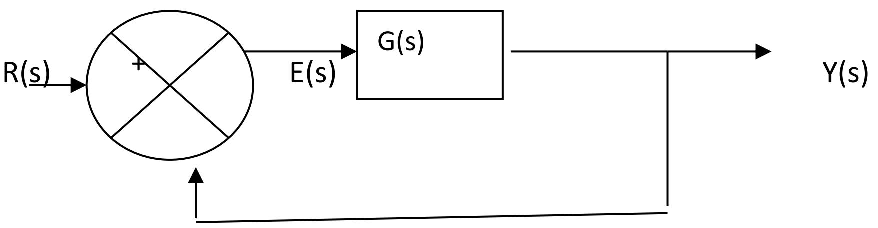


Fig 1.12: Unity feedback system

The steady state error of the system is written as

ess = $\lim_{t\to\infty} e(t)$ = $\lim_{s\to 0} sE(s)$ = $\lim_{s\to 0}[s(R(s) - y(s))]$

= $\lim_{s\to 0} s[R(s) - R(s)G(s)/(1 + G(s)H(s)]$

= $\lim_{s\to 0} s\,R(s)/(1 + G(s)$ (1)

Clearly ess depends on the characteristics of G(s). More specifically, we can show that ess depends on the number of poles that G(s) has at s = 0. This number is known as the **TYPE of the control system**. Or simply, **system Type**. We can show that the steady state error ess depends on the type of the control system. In general G(s) can be represented by:

G (s) = (K (1+T_1s)(1+T_2s)…(1+Tm_1 s+Tm_2 s^2))/(s^j (1+Ta s)(1+Tb s)…(1+Tn1 s+Tn2 s^2)) (2)

Where K and all T's are real constants. The system type represents order of the pole of G(s) at s=0. Thus the closed loop system having the forward path transfer function of above equation is type j, where j = 0, 1, 2… The following example illustrates the system type with reference to the form of G(s):

G(s) = K (1+0.5 s)/ ((s (1+ s) (1+2 s) (1+s+s^2)) **Type 1**

G(s) = K (1+ 2 s)/ (s^3) **Type 3**

Steady state error of a system with Step input: when input r(t) to the control system is a step function with magnitude R, R(s) = R/s. the steady state error is written from equation (1)

ess = $\lim_{s\to 0} s\,R(s)/(1 + G(s)$ = $\lim_{s\to 0} R/(1 + G(s)$ = R/ (1+ $\lim_{s\to 0} G(s)$)

For convenience, we define Kp = $\lim_{s\to} G(s)$

Hence ess = R/(1 + Kp) (3)

We can see from equation (3) that for ess to be zero Kp must be infinite. If G(s) is as shown in equation (2) then for Kp to be infinite j must be at least equal to unity, that is , G(s) must have at least one pole at s = 0. Therefore, we can summarize the steady state error due to step function input as follows:

IMPORTANT

Type 0 system: ess = R/ (1+ Kp) = constant.
Type 1 or higher system: ess = 0.

Kp is known as **position error constant**.

Steady State error with a Ramp function input: When the input is a ramp function with magnitude R, r(t) = R t where R is real constant, the Laplace transform of r(t) is R(s)

R(s) = R/s^2

The steady state error is ess = $\underset{s\to 0}{\text{limt}}\ R/(s + s\ G(s))$ = $\underset{s\to 0}{\text{limt}}\ R/(sG(s))$

We define the ramp error constant as Kv; where

Kv = $\underset{s\to 0}{\text{limt}}\, sG(s)$

Then ess = R/Kv

Hence for ess to be zero, Kv must be infinite. Using equation (2) we obtain

Kv = $\underset{s\to 0}{\text{limt}}\, s\ G(s)$ = $\underset{s\to 0}{\text{limt}}\, K$ / s^{j-1}

Thus, for Kv to be infinite, j must at least be equal to 2, or the system must be type 2 or higher. The following conclusions may be stated with regard to steady state error with ramp input.

Type 0 system: ess = ∞
Type 1 system: ess = R/Kv = constant
Type 2 or higher order: ess = 0

Steady state error with of system with Parabolic Input: When input is described by the standard parabolic form,

r(t) = R t^2 /2

The Laplace transform of r(t) = r/ s^3

The steady state error is ess = $\frac{R}{\underset{s\to 0\ G(s)}{\lim}\ s^2}$; defining the parabolic error constant as Ka

Ka = $\underset{s\to 0}{\lim}\ \ s2\ \ G(s)$; the steady state error becomes

ess = R/Ka

Following the pattern set with the step & ramp input, the steady state error due to parabolic input is zero if the system is 3 or greater. The following conclusions are made with regard to steady state error of a system with parabolic input.

Type 0 system: ess = ∞
Type 1 system: ess =∞
Type 2 system: ess = R/Ka = constant
Type 3 or higher order: ess = 0

Example: Consider a closed loop unity feedback system has the following transfer functions. The error constants and steady state errors are calculated for three basic inputs using the error constants;

(a) G(s) = $\frac{K(s+3.5)}{s(s+1.5)(s+0.5)}$, H(s) =1
For step input: step error constant Kp =∞, ess = R/(1+Kp) = 0
Ramp input: Kv = 4.2 K, ess = R/(4.2k)
Parabolic Input: ka = 0, ess = R/ka = ∞

(b) Let G(s) = 5(s+1)/((s^2 (s+12)(s+5))
We can calculate the error constants and steady state error for three basic inputs:
Step input: Kp = ∞ ; ess = R/(1+Kp) = 0
Ramp input: Kv = ∞ , ess= R/Kv = 0
Parabolic input: Ka = 1/12,ess = R/Ka = 12R.

1.10 Control System Components-sensors, transducers, servomotors, actuators, filters-modeling, transfer function:

1.10.1 Sensor: We can define a sensor as a device that converts a physical stimulus or input into a reliable output, which today would preferably be electronic, but which can also be communicated by other means such as visual and acoustic. The generic block diagram for a sensor is shown in fig 1.13 which highlights the role of a sensor as an interface between a control system and the physical world.

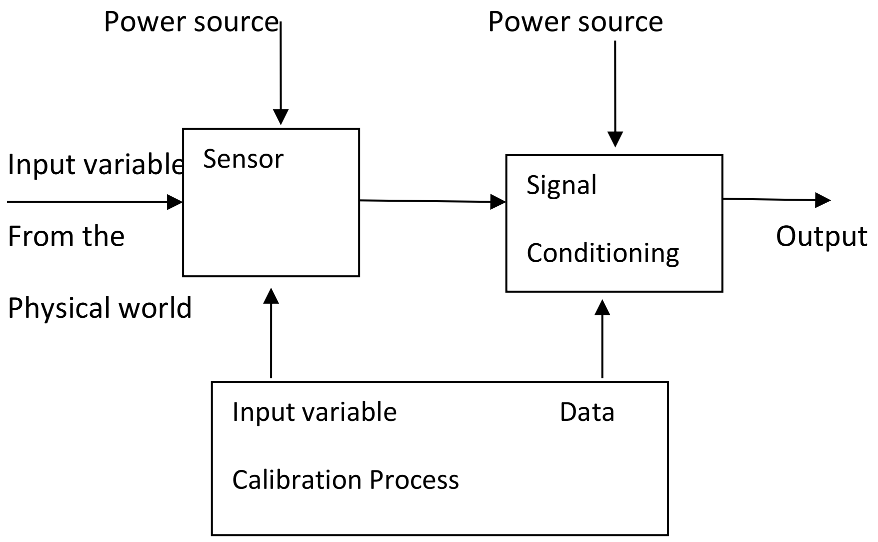


Fig: 1.13: Sensor Block Diagram

Sensors used in aircraft control system:

(a) Pitot-static sensor: for sensing Pitot & Static pressure.
(b) Temperature sensor: Thermo-couple, resistance based.
(c) Roll, Pitch, Yaw sensor: Mechanical, Laser Gyros.
(d) Acceleration sensors: Inertial navigation based.
(e) Velocity sensors: INS-GPS system based.
(f) Angle of attack sensors
(g) Angle of sideslip sensors.

Application of sensor in aircraft auto-pilot: This is shown in Fig 1.14

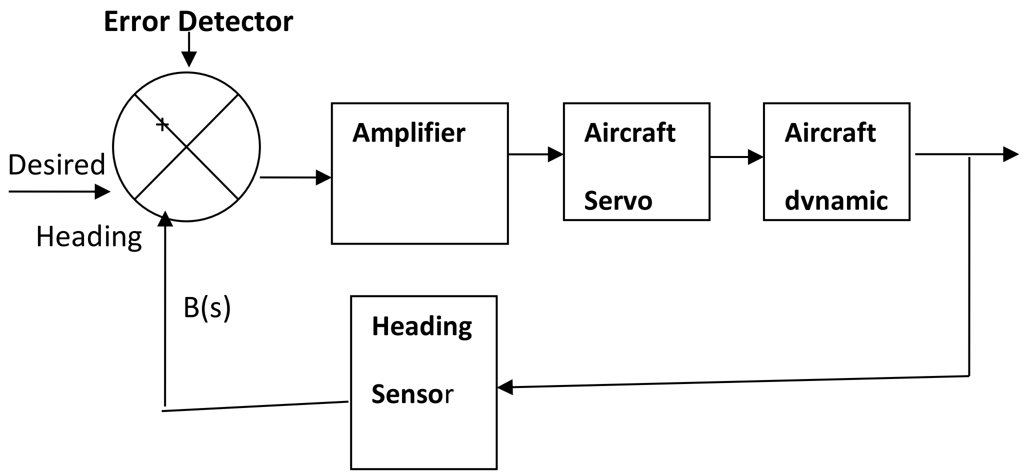


Fig 1.14: An overview of the Heading holding system

Mathematical Modeling and transfer function of sensors: Mathematical modeling of sensors can be derived by writing the differential equation governing input and output. Then taking the Laplace transform of both side of differential equation we can find the transfer function of the sensor. For example, Gyros are generally very accurate in low frequency measurements, but not so good in high frequency regions. So, we can model a gyro as a low pass filter, being

$H_{gyro}(s) = 1/(s + \omega_{br})$

The gyro break frequency, (above which the performance starts to decrease) is quite high. In fact, it is usually higher than any of the important frequencies of the aircraft. Therefore, gyro can often be simply modeled as H(s) =1. In other words, it can be assumed that the gyro is sufficiently accurate.

1.10.2 Transducer: A transducer is a device that transforms one form of energy into another. Transducers are generally made as small as possible, and the energy being transferred is small. Conversion between input and output is done quantitatively using a calibration process. Transducers use basic physical laws to measure physical parameters using sensing elements that is the part of transducer. The parameters measured in a servo control system are position and motion while parameters measured in process control systems are temperature, flow, level, pressure and others.

Examples of transducer: Potentiometer, LVDT (Linear variable differential transformers) tachometer, encoders.

Mathematical Modeling and transfer function of transducer. A potentiometer is an electromechanical transducer that coverts mechanical energy into electrical energy. The input to the device is in the form of a mechanical displacement. When a voltage is applied across fixed terminals, the output voltage, which is measured variable, is proportional to the input displacement as shown in fig 1.15. E is the applied voltage across fixed terminal. The output voltage is proportional to the shaft position θ(t).

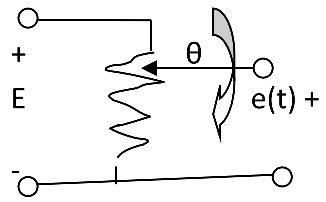


Fig 1.15: Potentiometer

Then, e (t) = K θ(t) ; where K is proportionality constant. Hence E(s) = K θ(s); or transfer Function is

$E(s)/\theta(s) = K$

Mathematical model and Transfer function of tachometer: Tachometer is electromechanical device that convert mechanical energy into electrical energy. The output voltage is proportional to the angular velocity of the input shaft. The dynamics of a tachometer can be represented by the equation

$e(t) = K_t\,(d\theta/dt) = K_t\,\omega(t)$

Where e(t) is the output voltage(t) is the rotor displacement in radians, ω(t) is the rotor velocity, K_t is tachometer constant in V/rad/sec. the value of K_t is given as a catalog parameter in volts per 1000 rpm. Transfer function of the tachometer is obtained by taking the Laplace transform of both sides.

$E(s)/\theta(s) = s\,K_t$

1.10.3 Servomotor: servomotors are widely used in control system as position controller. A DC servomotor is basically a torque transducer that converts electrical energy into mechanical energy. The torque developed on the motor is directly proportional to the field flux and armature current. The relationship among the torque developed, the flux ϕ and current i_a is

$T_m = K_m\,\phi\,i_a$

In addition to the voltage, the back emf, which is proportional to the shaft velocity, tends to oppose the current flow. The relationship between the back emf and shaft velocity is

$E_b = K_b\,\phi\,\omega_m$; where ω_m is the shaft velocity.

Mathematical modeling of a DC Motor: Mathematical model is shown in fig 1.16

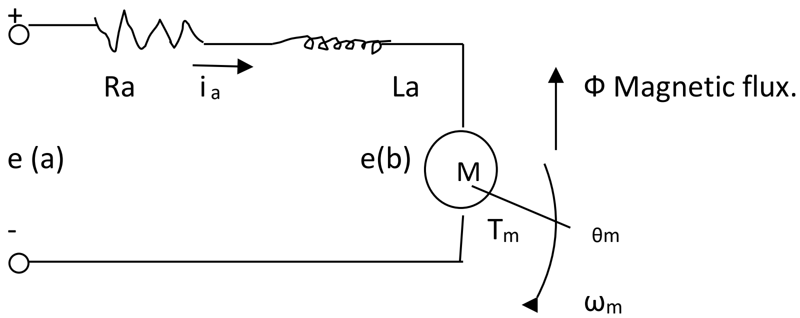


Fig: 1.16: Mathematical Model of Permanent Magnet Motor

$T_m(t) = K_i\, i_a(t)$

$e_a(t) = R_a\, i_a(t) + L_a\, d(i_a)/dt + e_b(t)$; $T_m(t) = K_i\, i_a(t)$

$e_b(t) = K_b\, d\Theta_m/dt = K_b\, \omega_m(t)$

$J_m\, d^2\Theta/dt^2 = T_m(t) - B_m\, d\Theta_m/dt$

Where $i_a(t)$ = armature current; L_a = armature inductance

$E_a(t)$ = applied voltage; $E_b(t)$ = back emf

$T_m(t)$ = Motor torque; K_i = Torque constant

K_b = Back emf constant; Φ = magnetic flux

$\omega_m(t)$ = rotor armature velocity; J_m = rotor inertia

B_m = Viscous friction force

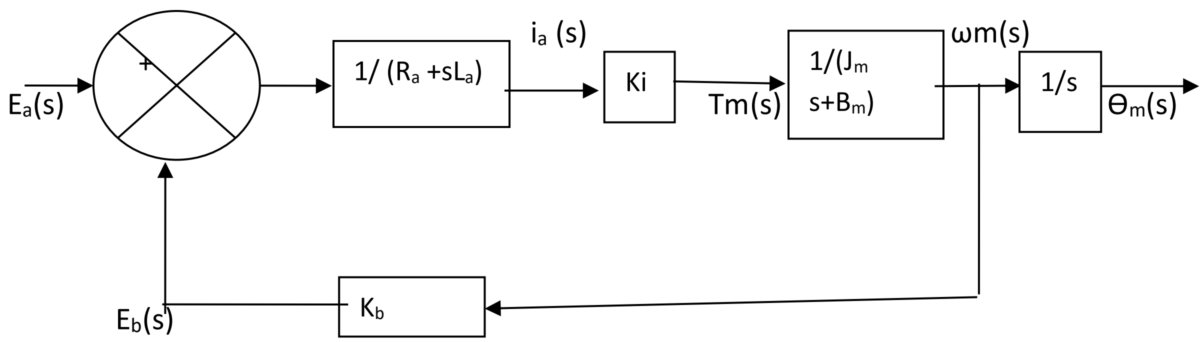


Fig: 1.17: transfer Function Block diagram

The transfer function between the motor displacement & the input voltage is given by (refer Fig 1.17):

θm (s)/E$_a$(s) = $\frac{K_i}{L_a J_m s^3+(R_a J_m+B_m L_a)s^2+(K_b K_i+R_a B_m)s}$

1.10.4 Actuators Function, Modeling and Transfer function: An example of a controller for an aircraft system is a hydraulic actuator used to move to the control surface. A control valve on the actuator is positioned by either a mechanical or electrical input, the control valve ports hydraulic fluid under pressure to the actuator, and the actuator piston moves until the control valve shuts off the hydraulic fluid. A hydraulic actuator is shown below in fig 1.18.

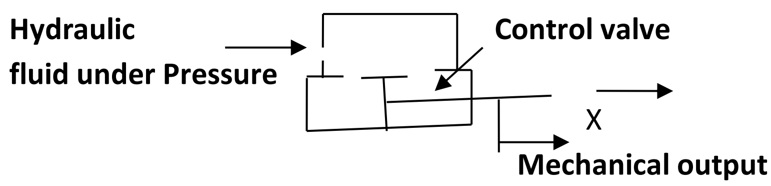


Fig 1.18: Hydraulic actuator

Clearly actuator piston cannot move instantaneously because it takes a finite time for the hydraulic fluid to move through the ports from the control valve. In response to a step unit, the resulting motion (x) of hydraulic actuator can be modeled as an exponential.

x (t) = Z (1-e^{-at})

, Where Z is the final displacement value of the actuator. Generalized transfer function of the actuator is: X(s)/E(s) = $\frac{a}{s+a}$

Where X(s) is the Laplace transform of the output & E(s) is Laplace transform of input. Block diagram of actuator with transfer function is shown in fig 1.19

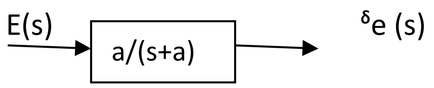
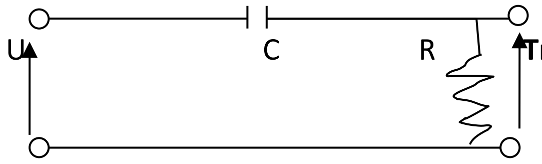


Fig 1.19: transfer function of a Actuator

1.10.5 Filters, Purpose, Modeling and Transfer Function: A powerful tool available to the control engineer is compensation filters. Compensation filters can various forms and are very affective in tailoring the aircraft response. They are of the following types:

(a) **Lead Compensator/High Frequency Filter: Purpose, Modeling and Transfer Function**: A lead compensator is used to quicken the system response by increasing natural frequency and/ or decreasing time constant. A lead compensator also increases the overall stability of the system. A simple lead compensator using simple RC network is shown below:



(b) **Lag Compensator/High Frequency Filter: Purpose, modeling and Transfer Function**. They are used to slow the system response by decreasing the natural frequency and/or increasing the time constant. They also tend to decrease the overall stability of the system. A simple lag network is shown in the figure below. They attenuate high frequencies like noise and disturbance.

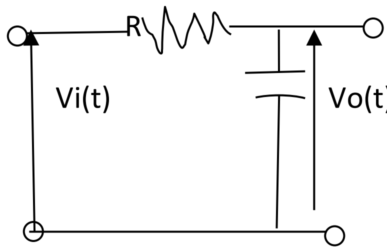


Fig: A simple Lag Network

(c) **Lead Lag filter**. The combined benefits of lead compensator and lag compensator may be realized using lead-lag compensation. Common use of lead-lag compensator is the attenuation of a specific frequency range (sometimes called notch filter). For example, an aircraft structural resonant frequency can be filtered out with a lead-lag compensator if a feedback sensor is erroneously affected by that frequency. Transfer function of a lead-lag compensator can be represented by:

TF $=\frac{bd(s+a)(s+c)}{ac(s+b)(s+d)}$ where a> b; a <c; c<d. (s+a)/(s+b) component corresponds to lag filter, s+c/s+d component represents the lead filter.

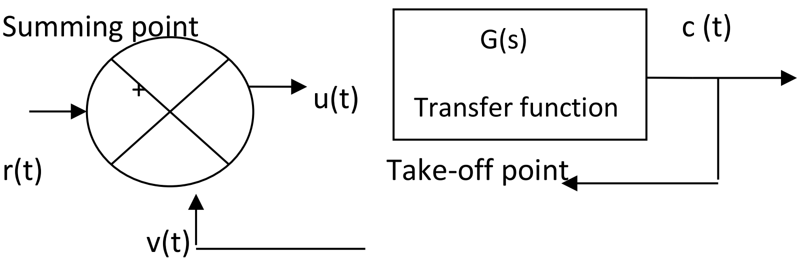
(d) **Washout Filter**. Another type of high pass filter which is used commonly in aircraft SAS is wash out filter. It is simply a case of the lead compensator where the zero is actually a differentiator. It has the transfer function as

TF $=\frac{Kw\ s}{s+b}$

Low frequency signals are attenuated, or washed out. Only changes in the input are passed through. This valuable for aircraft feedback control because feeding back a parameter such as roll rate with a wash out filter, the SAS would constantly oppose the roll rate and decrease the performance. The gain for high frequencies is determined by the corner frequency& the wash out filter gain Kw. Additionally, phase lead is added at higher frequencies.

1.11 Composition, reduction of Block diagrams of complex systems-rules and conventions:

There are four basic components of a block diagram. First there are blocks themselves, describing the relationship between input and output quantities through a transfer function. There are summing points where output parts of two or more blocks are added algebraically. Third component is a take-off point which represents the application of the total output from that point as input to some other block. Finally, diagrams contain arrows, indicating unidirectional flow of signal in these diagrams. This will be clear from the following fig.

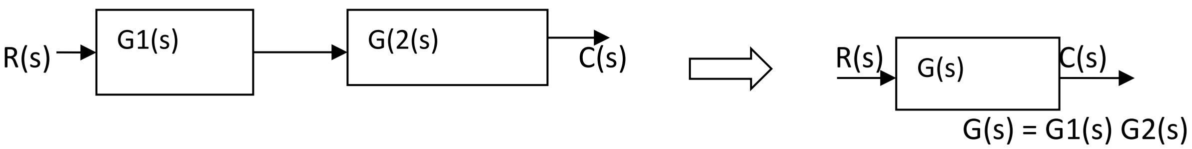


u(t) = r(t) – v(t)

C(s) = G(s) U(s)

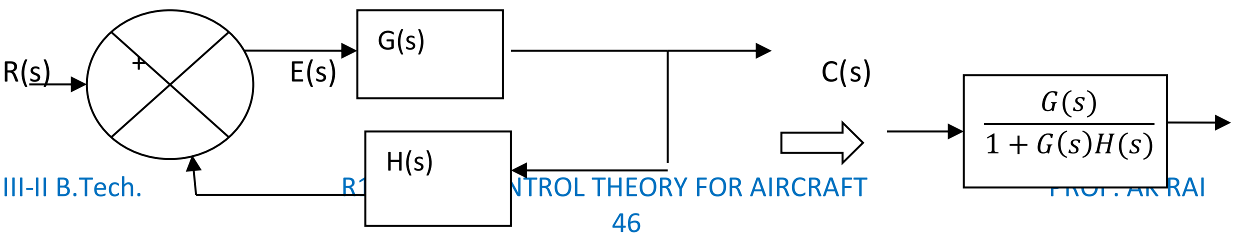
Rules of Block Diagram Algebra:

1. **Combining cascade Blocks**: Blocks connected in cascade can be replaced by a simple block with transfer function equal to the product of the respective transfer function. This is shown below:



This is valid only if no loading effect on first block due to second block.

2. **Elimination of a feedback control**: Let G(s) be the transfer function in the forward path, H(s) is transfer function of feedback path. Overall transfer function is shown in the fig below.



C(s) H(s)

Example: Determine the overall transfer function of the system shown in Fig 1.16.1 below.

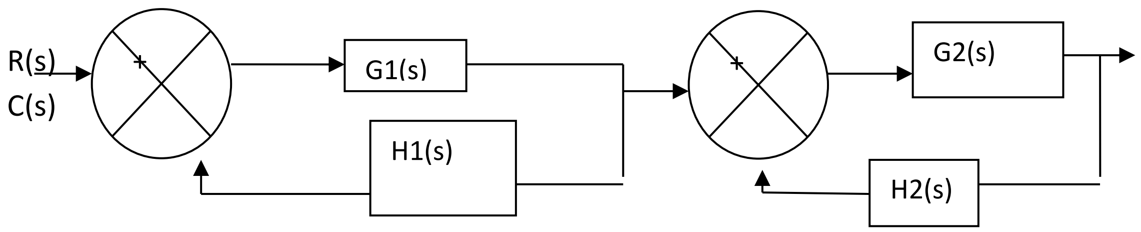
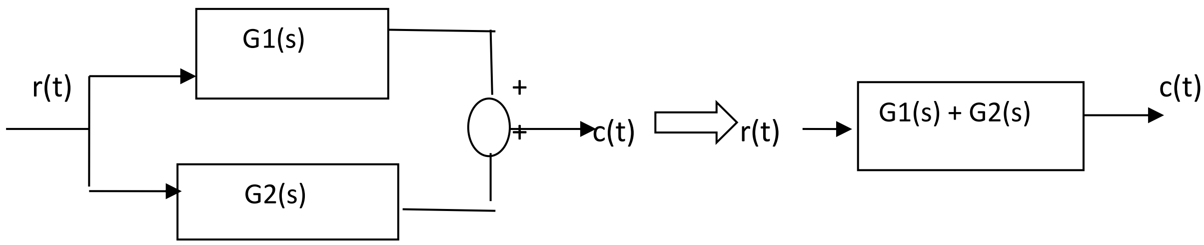


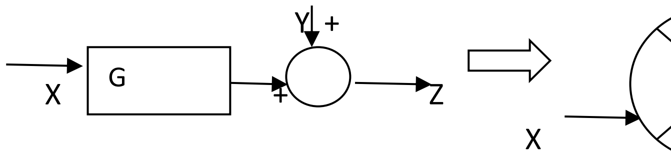
Fig 1.16.1

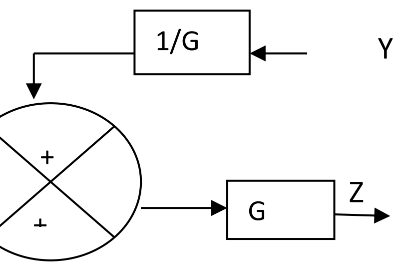
Ans: $\frac{G1(s)G2(s)}{(1+G1(s)(H1(s))(1+G2(s)H2(s))}$

3. **Parallel Block**: Blocks are said to be parallel if they have common input & the overall output is sum of the individual outputs. The transfer function of the parallel combination is simply the sum of the transfer function of the parallel blocks. This is shown below:

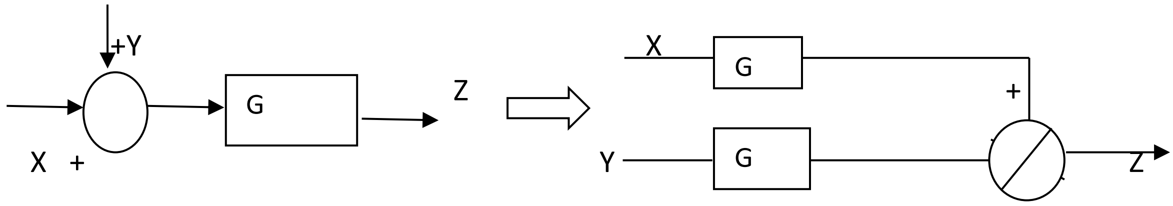


4. **Moving a Summing Point ahead of a block**:

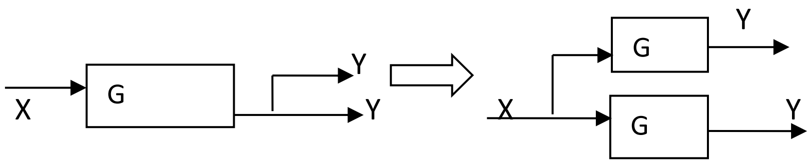




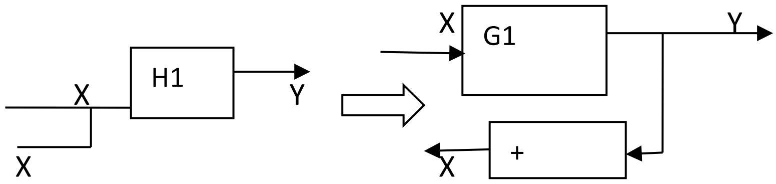
5. **Moving a Summing point behind a Block**:



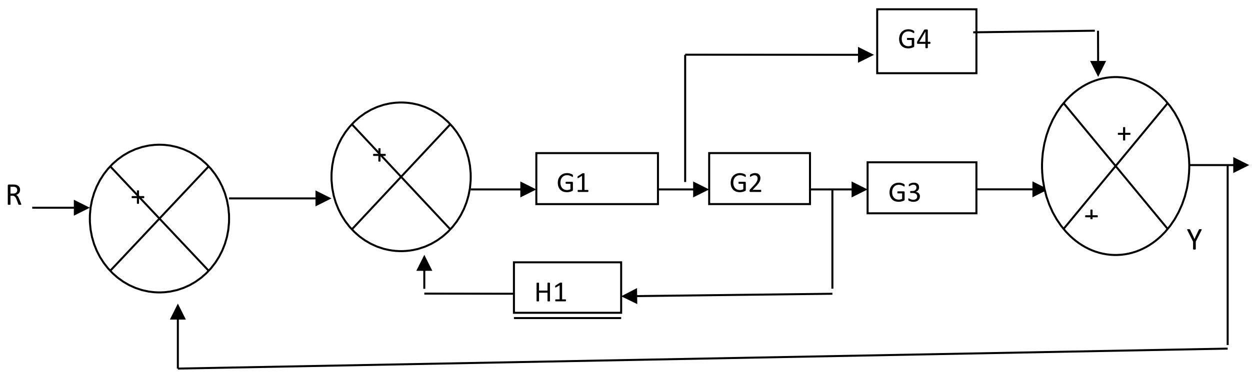
6. **Moving a pick-off point ahead of a Block**:



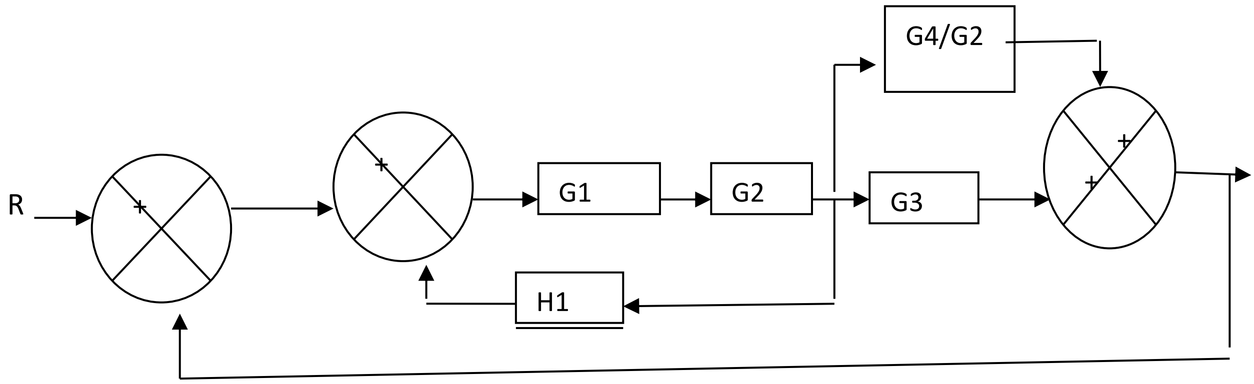
7. **Moving a Pick-off Point behind a Block**:



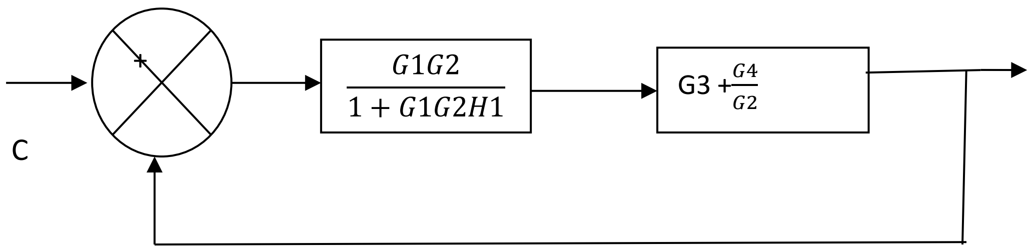
Examples of Block Reduction: Find Y/R for the following blocks**:**



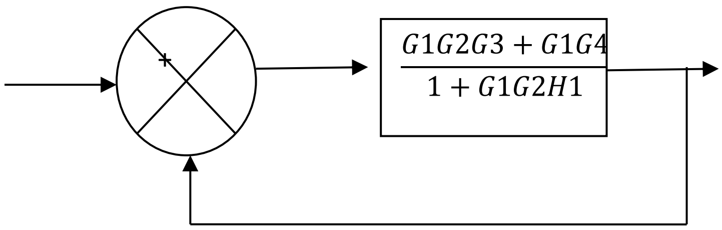
Solution: Above block can be reduced as follows: We shift pick-off point after block G2.



This can be further reduced to:



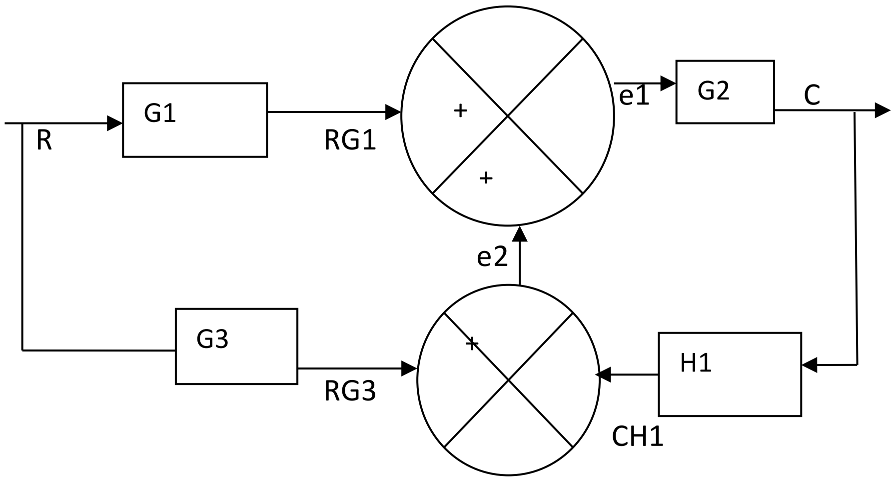
This can be reduced as follows:



Hence final transfer function is:

C(s)/R(s) = $\frac{G1G4+G1G2G3}{1+G1G2H1+G1G2G3+G1G4}$ Ans.

Problem 2: Reduce the block diagram & find C/R



Solution: e2 = RG3 – CH1

e1= RG1 +e2 = RG1 +RG3-H1C ;

C=e1 G2 (substituting value of e1)

C = (RG1+RG3 –H1C) G2

RG1G2 + RG3G2 = G2H1C +C;

Hence, C/R = $\frac{G1G2+G3G2}{1+H1G2}$

UNIT-II

Performance-Time, Frequency and S-domain Description.

Index

2.1 Control System Performance-Time domain description- Output Response to Control Inputs-Impulse response, characteristic parameters-relation to system parameters.

2.**1.1 Time domain description of a first order system**: Consider the first order system shown in fig 2.1 below. Physically the system may represent an RC circuit or the like. The input output relationship is given by

C(s)/R(s) = 1/ (Ts+1) (1)

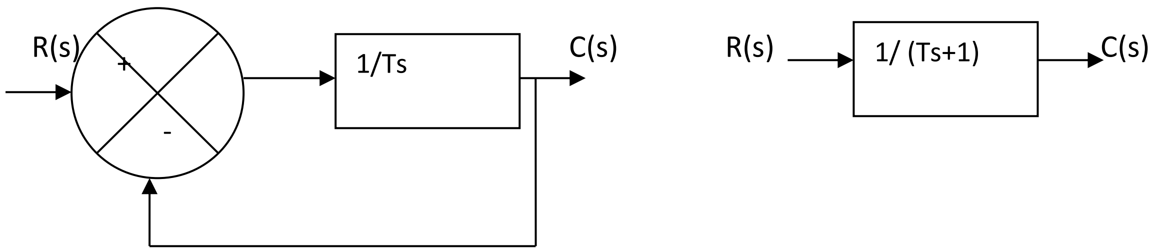


Fig 2. 1: First order System

In the following section we will analyze the system response to such input as unit-step, unit-ramp, and a unit impulse function. The initial conditions are assumed to be zero.

(a) Unit Step Response of First Order System: Since the Laplace transform of the unit step function is 1/s, substituting R(s) = 1/s into equation (1) we obtain

C(s) = $\frac{1}{(Ts+1)(s)}$, Expanding C(s) into partial fraction gives

C(s) = $\frac{1}{s}$- $\frac{T}{Ts+1}$ = = $\frac{1}{s}$- $\frac{1}{s+1/T}$ (2)

Taking the inverse Laplace transform of equation (2), we obtain

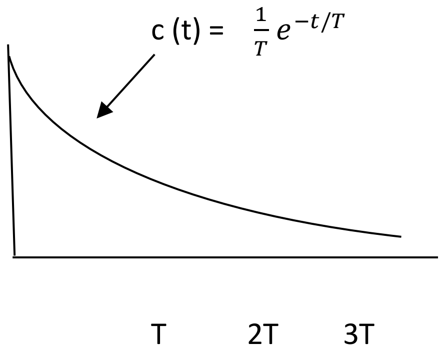
c (t) = 1- $e^{-t/T}$, for t >= 0 (3)

Equation (3) states that initially the output c(t) is zero and finally it becomes unity. One important characteristic of such exponential response curve c(t) is that at t= T , the value of c(t) is 0.632 or the response c(t) has reached 63.2% of its total change. This may be easily seen by substituting the t=T in c(t). That is c(T) = 1- 1/e = 0.632. Note that smaller the time constant T, the faster the system. Another important characteristic of the exponential response curve is that the slope of the tangent line at t = 0 is 1/T. Since dc/dt at t=0 equals to 1/T ($e^{-t/T}$) at t=0 equals 1/T. The output would reach the final value at t= T, if it maintained its initial speed of response. We see that slope of the response curve c(t) decreases monotonically from 1/T at t=0 to zero at t=∞. In one time constant, the exponential response curve has gone from 0 to 63.2% of the final value. In two time constant, the response reaches 86.5% and at t=3T it reaches 95% of the final value.

(b) **Unit-Impulse Response of the First Order System**: For the unit-impulse input, R(s) = 1 and the output of the system of fig 2.1 can be obtained as

C(s) = $\frac{1}{Ts+1}$; Inverse Laplace transform of the above equation gives

c (t) = $\frac{1}{T}e^{-t/T}$ for t>= 0



(a) Time Domain Description of a Second Order System: Consider a typical second order system shown in fig 2.2(a) below. Servo system consists of a proportional controller and load elements (inertia and viscous friction element). Suppose we wish to control the output c in accordance with the input r.

The equation for the load element is:

$J\ddot{C}$ + $B\dot{C}$ = T Where T is the torque produced by the proportional voltage gain K. By taking the Laplace transform of both side of the last equation, assuming zero initial condition, we obtain

(Js^2 + Bs) C(s) = T(s) ; So the transfer function between C(s) and T(s) is

C(s)/T(s) = $\frac{1}{s(Js+B)}$; By using this transfer function; Fig 2.2(a) can be redrawn as Fig 2.2(b),

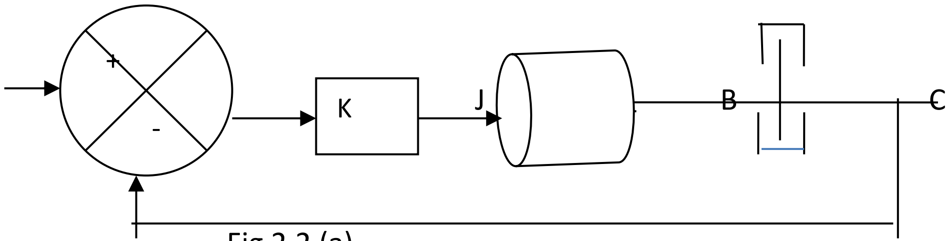


Fig 2.2 (a)

the closed loop transfer is then obtained as

$C(s)/R(s) = K/ (Js^2 + Bs + K) = (K/J)/(s^2 + (B/J) s+(K/J))$

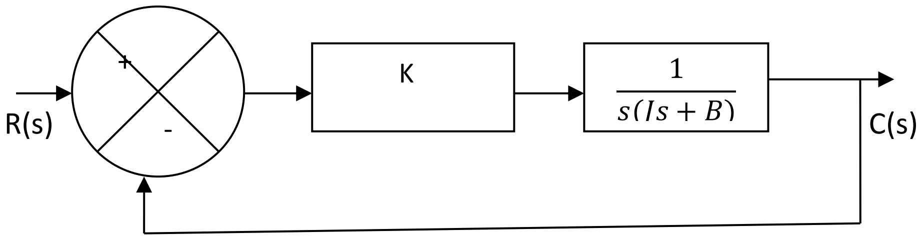


Fig 2.2 (b)

Step Response of a second order System: The closed loop transfer function of the system is

$C(s)/R(s) = K/(Js^2 + Bs + K) = (K/J)/(s^2 + (B/J) s+(K/J)) = \omega_n^2/(s^2 + 2\zeta \omega_n s + \omega n^2)$

Where $K/J = \omega n^2$; $B/J = + 2\zeta \omega_n = 2\sigma$

Where σ is called the attenuation; ω_n is called un damped natural frequency, and ζ is called the damping ratio of the system. These are called **system parameters**. In terms of ζ and ω_n , the given system can be modeled to the system shown in fig 2.3 below. And closed loop transfer function c(s)/R(s) can be written as

$C(s)/R(s) = \omega_n^2/ (s^2 + 2\zeta \omega_n s + \omega n^2)$

Characteristic equation is: $s^2 + 2\zeta \omega_n s + \omega n^2 = 0$

This form is called the standard form of the second order system.

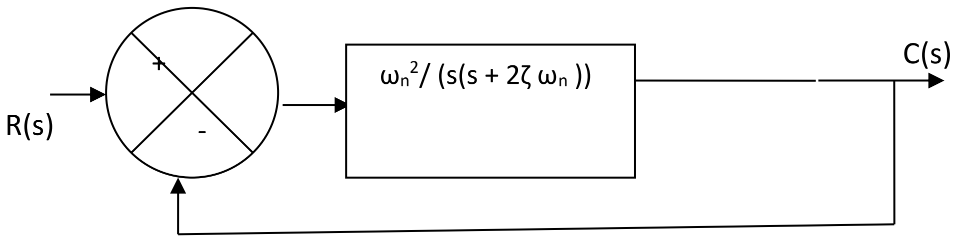


Fig 2.3: Second Order System

The dynamic behavior of the second order system can be described in terms of two parameters ζ and ω_n. If $0 < \zeta < 1$, the closed loop poles are complex and lie in the left-half of s-plane. The system is called under damped, and the transient response is oscillatory. If ζ

=0, the transient response does not die out. If $\zeta = 1$ the system is called critically damped. Over damped system corresponds to $\zeta > 1$. Let us consider the unit step response for all the three cases: under damped, critically damped ($\zeta = 1$) and over damped ($\zeta > 1$).

Case 1: Under damped ($0 < \zeta < 1$). In this case C(s)/R(s) can be written as:

$C(s)/R(s) = \omega_n{}^2 / ((s+ \zeta \omega_n + j\omega_d)(s+ \zeta \omega_n - j\omega_d))^2$

Where $\omega_d = \omega_n \sqrt{1-\zeta^2}$. The frequency ω_d is called damped natural frequency. For a unit step input, C(s) can be written as

$C(s)= \omega n^2 / (s^2 + 2\zeta \omega_n s + \omega n^2)(s)$

The inverse Laplace transform can be obtained easily if C(s) is written in the following form;

$C(s) = \frac{1}{s} - (s+ 2\zeta \omega_n) / (s^2 + 2\zeta \omega_n s + \omega n^2)$

$= = \frac{1}{s} - (s+\zeta \omega_n) / ((s+\zeta \omega_n)^2 + \omega_d^2) - \zeta \omega_n / ((s+\zeta \omega_n)^2 + \omega_d^2)$

We know that L^{-1} of $(s+\zeta \omega_n) / ((s+\zeta \omega_n)^2 + \omega_d^2) = e^{-\zeta\omega nt}$ Cos (ω_d t)

and L^{-1} of $\omega_d / ((s+\zeta \omega_n)^2 + \omega_d^2) = e^{-\zeta\, \omega nt}$ Sin(ω_d t)

Hence the inverse Laplace transform of C(s) can be obtained as

$c(t) = 1- e^{-\zeta \omega n t} (\mathrm{Cos}\, \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \mathrm{Sin}(\omega_d t)$

$=1-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta \omega n t} \mathrm{Sin} (\omega_d t + \tan^{-1} \sqrt{1-\zeta^2} / \zeta)$

If the damping ratio is zero, the response becomes un damped & oscillations continue indefinitely. In this case

$C (t) = 1- \cos (\omega_n t)$.

Thus, ωn represents un- damped natural frequency of the system.

Critically damped case (ξ =1). If the two poles are equal, the system is said to be critically damped one.

For a unit step input R(s) = 1/s & C(s) can be written as

$C(s) = \omega_n{}^2/((s+\omega_n)^2 s)$

Hence, $c(t) = 1- e^{-\zeta \omega n t} (1+ \omega_n t)$ for t>= 0

Over damped case: (**ξ >1).** Two poles are negative & unequal.

C(s) = $\omega_n{}^2$ / s ((s+ ζ ω_n + ω_n $\sqrt{1-\zeta^2}$) (s+ ζ ω_n - ω_n $\sqrt{1-\zeta^2}$))

C (t) = (ω_n/(2 $\sqrt{1-\zeta^2}$)) ($\frac{e^{-s1t}}{s1}$ - $\frac{e^{-s2t}}{s2}$)

Where s1 = (ζ + $\sqrt{\zeta^2-1)\omega n}$, s2= (ζ- $\sqrt{\zeta^2-1)\omega n}$)

Thus, the response c (t) includes two decaying exponential terms.

Impulse Response: Suppose input to a control system whose transfer function is G(s) is impulse input δ (t). In this case R(s) = 1,

Hence output c(s) = G(s); hence c (t) = Laplace inverse of G(s). Hence another method of defining transfer function is: Transfer function of a control system is Laplace transform of unit impulse response.

Characteristic Parameters & its relation to system parameters: For a second order system, system parameters are natural frequency ω_n and damping constant ξ. These are related to characteristic parameters which are defined as following (time response characteristic parameters): Refer Fig 2.4

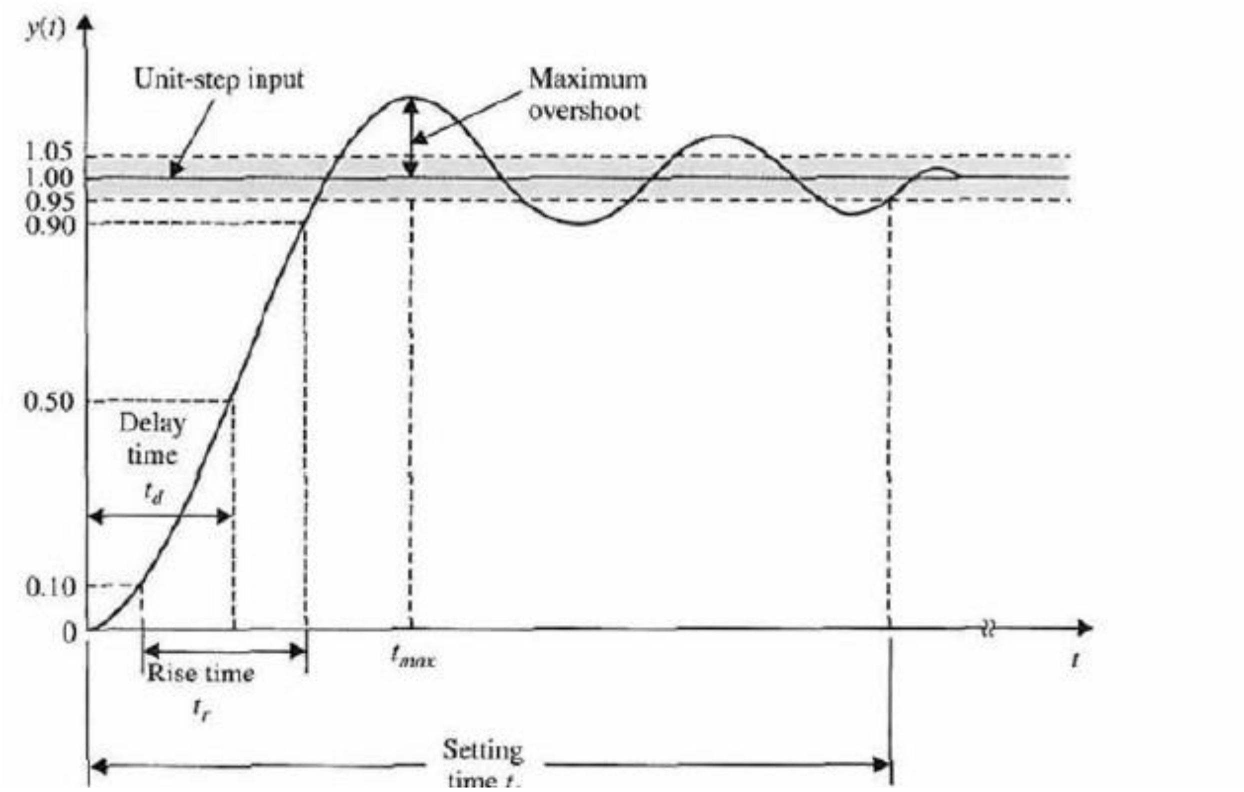


Fig: 2.4: Characteristic Parameters of second order System with unit step input.

Characteristic Parameters of second order System:

(i) Delay time, t_d.

(ii) Rise time, t_r.
(iii) Peak time, t_p.
(iv) Maximum overshoots, M_p
(v)Settling time, t_s.
(vi) Steady state error.

These parameters are defined as follows:

(a) Delay time t_d: The delay time is the time required for the response to half the final value for the first time. The delay time is related to system parameters for a second order system by:

$t_d \cong (1.\ 0.7\xi)/\omega n$; $0< \xi < 1.0$
We can obtain a better approximation by using a second order equation
$t_d \cong (1.1+ 0.25\xi + 0.469\xi^2)/\omega_n$

(b) Rise time t_r: The rise time is the time required for the response to rise from 10% to90% of its final value.
Rise time $t_r \cong (0.8+2.5\zeta)/ \omega_n$

(c)Peak time Tp: The peak time Tp is the time required for the response to reach the first peak of the overshoot.

(d)Maximum Overshoot, Mp: The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by:
Maximum percent overshoot = 100 *(c (t) – c (∞))/c (∞). The amount of maximum (percent) overshoot indicates the relative stability of the system.
% Maximum overshoot = 100 $e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$

(e) Settling time ts: The settling time ts is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).
Settling time for 5%
$t_s \cong 3.2/ (\zeta\ \omega_n)$ for $0< \zeta < 0.69$
$t_s \cong 4.5\zeta/\omega_n$; $\xi > 0.69$

(f) Steady state error for a unity feedback system is defined as the difference between output and input as time approaches infinity. We have already discussed steady state error in Unit-I.

Impulse response of a second order System: For a unit-impulse input r(t) corresponding Laplace transform is unity, or R(s) = 1. The unit impulse response of the second order system is

$C(s) = \omega n^2 / (s^2 + 2\zeta\, \omega_n\, s + \omega n^2)$

The inverse Laplace transform of this equation yields the solution c (t) as follows:

$c(t) = (\omega n/\sqrt{1-\zeta^2})\, e^{-\zeta \omega n t}\ \mathrm{Sin}\,(\omega n\sqrt{1-\zeta^2}\ t)$, for t>=0

For $\zeta = 1$

$c(t) = \omega_n^2\, t\ e^{-\zeta \omega n t}$

2.2 Laplace Transform. In order to transform a given function of time f(t) into its corresponding Laplace transform first multiply f(t) by e^{-st} , s being a complex number(s=σ+jω). Integrating this product w.r.t time with limits as zero to infinity. This integration results in Laplace transform of f(t), which is denoted by F(s) or ʆ f(t). The mathematical expression for Laplace transform is

ʆ f(t) = F(s) t>=0

Or $F(s) = \int_0^{\infty} f(t).e^{-st}dt$

The time function f(t) is obtained from the Laplace transform by a process called inverse Laplace transformation and denoted by L -1 thus

$ʆ^{-1}$ [L f(t)] = $ʆ^{-1}$ [F(s)]=f(t)

The time function f(t) and its Laplace transform F(s) are a transform pair.
Table 2.1 below gives transform pairs of some commonly used functions and Laplace transform pairs for some functions will be derived.

Table 2.1 Table of Laplace transform pairs

S. No	f(t)	F(s)
1	δ(t) unit impulse at t=0	1
2	u(t) unit step at t=0	1/s
3	u(t-T) unit step at t=T	$\frac{1}{s}e^{-sT}$
4	T	$\frac{1}{s^2}$
5	$\frac{t^2}{2}$	$\frac{1}{s^3}$
6	t^n	$\frac{!n}{s^n}$

7	e^{-at}	$\frac{1}{s+a}$
8	e^{at}	$\frac{1}{s-a}$
9	te^{-at}	$\frac{1}{(s+a)^2}$
10	te^{at}	$\frac{1}{(s-a)^2}$
11	$t^n e^{-at}$	$\frac{!n}{(s+a)^{n+1}}$
12	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
13	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
14	$e^{-\alpha t} \sin \omega t$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
15	$e^{-\alpha t} \cos \omega t$	$\frac{(s+\alpha)}{(s+\alpha)^2+\omega^2}$
16	$\sinh \alpha t$	$\frac{\alpha}{s^2-\alpha^2}$
17	$\cosh \alpha t$	$\frac{s}{s^2-\alpha^2}$

2.2.1 **Derivation of Laplace Transform**.

(a) Laplace transform of e^{at}

$\mathcal{L}\ e^{at} = \int_0^\infty e^{at}\, e^{-st} dt = \int_0^\infty e^{(s-at)} dt = \frac{1}{(s-a)}$

Inverse Laplace transform of 1/(s-a) is therefore

$\mathcal{L}^{-1}$ [1/(s-a)] = e^{at} (2.1)

(b) In the function f(t)= e^{at} put a=0;

Hence e^{at} = e^{0t} = 1. Hence, f(t)=1

Therefore, $\mathcal{L}$ [1]= 1/(s-0)

Or $\mathcal{L}$ [1]= 1/s

and $\mathcal{L}^{-1}$ [1/s] =1

(c) In the function f(t) = e^{at} put a= jω

e^{at} = $e^{j\omega t}$. hence f(t) = $e^{j\omega t}$

Therefore, $\mathcal{L}$ ($e^{j\omega t}$) = $\frac{1}{(s-j\omega)}$

$e^{j\omega t}$ = cosωt +j sin ωt

Hence, ʆ [(cosωt +j sin ωt)] = $\frac{1}{(s-j\omega)} = \frac{s+j\omega}{(s^2+\omega^2)}$

Separating into real and imaginary parts,

ʆ cosωt = $\frac{s}{s^2+\omega^2}$

ʆ sinωt = $\frac{\omega}{s^2+\omega^2}$

(d) In the function f(t) = e^{at} put a= $(\alpha - j\omega)$

Hence, e^{at} = $e^{(-\alpha+j\omega)t}$

Therefore, f(t)= $e^{(-\alpha+j\omega)t}$

Hence, using equation (2.1)

ʆ $e^{(-\alpha+j\omega)t} = \frac{1}{s-(-\alpha+j\omega)} = \frac{1}{(s+\alpha)-j\omega}$

Now since $e^{(-\alpha+j\omega)t} = e^{-at}$ (cosωt +j sin ωt)

ʆ e^{-at} (cosωt +j sin ωt) = $\frac{1}{(s+\alpha)-j\omega} = \frac{(s+\alpha)+j\omega}{((s+\alpha)^2+\omega^2)}$

Separating into real and imaginary parts

ʆ e^{-at} . cosωt = $\frac{(s+\alpha)}{(s+\alpha)^2+\omega^2}$

ʆ e^{-at} . sinωt = $\frac{\omega}{(s+\alpha)^2+\omega^2}$

ʆ $^{-1}\left[\frac{(s+\alpha)}{(s+\alpha)^2+\omega^2}\right] = e^{-at}$. cosωt

ʆ $^{-1}\left[\frac{\omega}{(s+\alpha)^2+\omega^2}\right] = e^{-at}$. sinωt

2.2.2 Basic Laplace Transform theorem.

(a) Laplace Transform of linear combination

ʆ [a f_1(t)+b f_2(t)] = aF_1(s) + bF_2(s)

Where, f_1(t), f_2(t) are functions of time and a, b are constants

(b) If the Laplace transform of f(t) is F(s), then

(i) ʆ $\frac{df(t)}{dt}$ = [s F(s) – f(0 +)]

(ii) ʆ $\frac{d^2f(t)}{dt^2}$= [s^2 F(s) -s f(0 +)-f'(0 +)]

(iii) ʆ $\frac{d^3f(t)}{dt^3}$= [s^3 F(s) -s^2f(0 +)- s f'(0 +)-f''(0 +)]

Where, f(0 +), f'(0 +), f''(0+)... are the values of f(t), $\frac{df(t)}{dt}$, $\frac{d^2f(t)}{dt^2}$... at t=(0 +)

(c) If the Laplace transform of f(t) is F(s), then

$$\mathcal{L} \int f(t) = [\frac{F(s)}{s} + \frac{f^{-1}(0+)}{s}]$$

$$\mathcal{L} \iint f(t) = [\frac{F(s)}{s^2} + \frac{f^{-1}(0+)}{s^2} + \frac{f^{-2}(0+)}{s}]$$

$$\mathcal{L} \iiint f(t) = [\frac{F(s)}{s^3} + \frac{f^{-1}(0+)}{s^3} + \frac{f^{-2}(0+)}{s^2} + \frac{f^{-1}(0+)}{s}]$$

where $f^{-1}(0+)$, $f^{-2}(0+)$, $f^{-3}(0+)$ ... are the values of $\int f(t)$, $\iint f(t)$, $\iiint f(t)$ at t= (0 +).

(d) If the Laplace transform of f(t) is F(s), then

$$\mathcal{L}\ e^{-at} f(t) = F(s+a)$$

(e) If the Laplace transform of f(t) is F(s), then

$$L\ t\ f(t) = -\frac{d}{ds}F(s)$$

(f) Initial value theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$$

(g) **Final value theorem**

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$$

The final Value theorem gives the final value (t→ ∞) of a time function using its Laplace transform and such very useful in the analysis of control systems. However, if the denominator of s F(s) has any root having real part as zero or positive, then the final value theorem is not valid.

2.2.3 **Application of Laplace Transform to Solution of Differential Equations**: A second order differential equation can be written as:

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = f(t)$$

Linear ordinary differential equations can be solved by the Laplace transform method with the aid of the theorems on Laplace transform given in section 2.2.2.

The procedure is outlined as follows:

(a) transform the differential equation into s-domain by Laplace transform using the Laplace transform table.

(b) Manipulate the transformed algebraic equation and solve for the output variable.

(c) Obtain the inverse Laplace transform from the Laplace transform table.

Example: consider the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\,\frac{dy(t)}{dt} + 2y(t) = 5\,u(t)$$

Where u(t) is unit step function. The initial conditions are y(0)=-1 and $y^1(0)$= 2. O solve the differential equation, we first take the Laplace transform on both side of the equation.

$s^2\, y(s) - s\, y(0) - y^{(1)}(0) + 3s\, Y(s) - 3\, y(0) + 2\, Y(s) = 5/s$

Substituting the values of initial conditions and solving for Y(s), we get

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)}$$

Converting into partial fraction we get

$$Y(s) = \frac{5}{2s} - \frac{5}{s+1} + \frac{3}{2(s+2)}$$

Taking inverse Laplace transform

$$y(t) = \frac{5}{2} - 5e^{-t} + \frac{3}{2}e^{-2t} \qquad \text{for } t \geq 0$$

2.2.4. **Partial Fraction Decomposition of transfer functions-significance**: Partial fraction decomposition helps in finding the out response for a given input. When the Laplace transform solution of a differential equation is a rational function is s, it can be written as

$G(s) = \frac{Q(s)}{P(s)}$; where P(s) and Q(s) are polynomials of s. it is assumed that order of P(s) is greater than that of Q(s). If G(s) has simple poles, it can be written as

$$G(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s+s_1)(s+s_2)\ldots(s+s_n)} \quad ; \text{where} \quad s_1 \neq s_2 \neq s_n$$

Applying the partial-fraction expansion, above equation can be written as

$$G(s) = \frac{K_{s1}}{s+s_1} + \frac{K_{s2}}{s+s_2} + \ldots + \frac{K_{sn}}{s+s_n}$$

The coefficients K_{si} (i=1,2,…,n) is determined by multiplying both sides of the equation by the factor $(s+s_i)$ and then setting s equal to $-s_i$. To find the coefficient, for instance, we multiply both sides by $(s+s_1)$ and let $s=-s_1$. Thus

$Ks_1=[(s+s_1)Q(s)/P(s)]|s=-s_1 \quad = \frac{Q(-s1)}{(s2-s1)(s3-s1)\ldots(sn-s1)}$

Example: consider the function

$$G(s)= \frac{5s+3}{(s+1)(s+2)(s+3)}$$

Which is written in the partial fraction form as

$$G(s)= \frac{K_{-1}}{s+1} + \frac{K_{-2}}{s+2} + \frac{K_{-3}}{s+3}$$

The coefficients K_{-1}, K_{-2} , K_{-3} are determined as follows

$K_{-1}= [(s+1)\, G(s)]|_{s=-1} = \frac{5(-1)+3}{(2-1)(3-1)} = -1$

$K_{-2}= [(s+2)\, G(s)]|_{s=-2} = \frac{5(-1)+3}{(2-1)(3-1)} = 7$

$K_{-3}= [(s+1)\, G(s)]|_{s=-3} = \frac{5(-1)+3}{(2-1)(3-1)} = -6$

Hence G(s) can be written as:

$$G(s)= \frac{-1}{s+1} + \frac{7}{s+2} - \frac{6}{s+3}$$

Partial Fraction when G(s) has multiple poles:

Let $G(s) = (s^2+2s+3)/(s+1)^3$

$G(s) = \frac{b1}{s+1} + \frac{b2}{s+2} + b3/(s+1)^3$

We can determine b1, b2, b3 by comparing the coefficient of s2, s1, so of both sides by multiplying both sides by $(s+1)^3$.

2.2.5 Poles and Zeros. The transfer function of a linear control system can be expressed in the form of a quotient of polynomials in the following form:

$$G(s)=\frac{A(s)}{B(s)} = \frac{a_0s^n+a_1s^{n-1}+a_2s^{n-2}+\cdots+a_n}{b_0s^m+b_1s^{m-1}+b_2s^{m-2}+\cdots+b_m}$$

The numerator and the denominator can be factored into n and m terms respectively, with such a factorization the above expression for the transfer function can be expressed as,

$G(s)=\frac{A(s)}{B(s)} = \frac{K(s-s_1)(s-s_2)\ldots(s-s_n)}{(s-s_a)(s-s_b)\ldots(s-s_m)}$ (2.5)

Where K = $\frac{a_0}{b_0}$ is known as the gain factor of the transfer function.

In the transfer function expression (2.5), if the numerator is equated to zero, then n roots of the equation are $s_1, s_2, \ldots s_n$. and whereas equating the denominator to zero, the m roots can be determined as $s_a, s_b, \ldots, s_m$.

Poles of the transfer function: in the transfer function expression (2.5), if s is put equal to $s_a, s_b, \ldots, s_m$ it is noted that the value of the transfer function in infinite, hence $s_a, s_b, \ldots, s_m$ are called the Poles of the transfer function.

Zeros of the transfer function: in the transfer function expression, if s is put equal to $s_1, s_2, \ldots s_n$ it is noted that the value of the transfer function is zero, hence $s_1, s_2, \ldots s_n$ are called the Zeros of the transfer function.

The zeros $s_1, s_2, \ldots s_n$ or poles $s_a, s_b, \ldots, s_m$ are either real or complex and the complex poles or zeros appear in conjugate pairs. It is possible that either poles or zeros may coincide. Such poles or zeros are called multiples poles or multiple zeros.

The graphical symbols for the pole is X and for a zero is 0. The said symbols are used when poles and zeros are to be shown on a real and imaginary axes(s-plane).

Consider the transfer function

$G(s) = \frac{(s+2)(s+4)}{s(s+3)(s+5)(s+2-j4)(s+2+j4)}$

For the above transfer function, the poles are (1) $s_a = 0$, (2) $s_b = -3$, (3) $s_c = (-2+j4)$,

(4) $s_d = (-2-j4)$ and (5) $s_c = -5$. The zeros are at (1) $s_1 = -2$, (2) $s_2 = -4$.

The pole-zero locations are plotted s-plane are shown in fig 2.5 below:

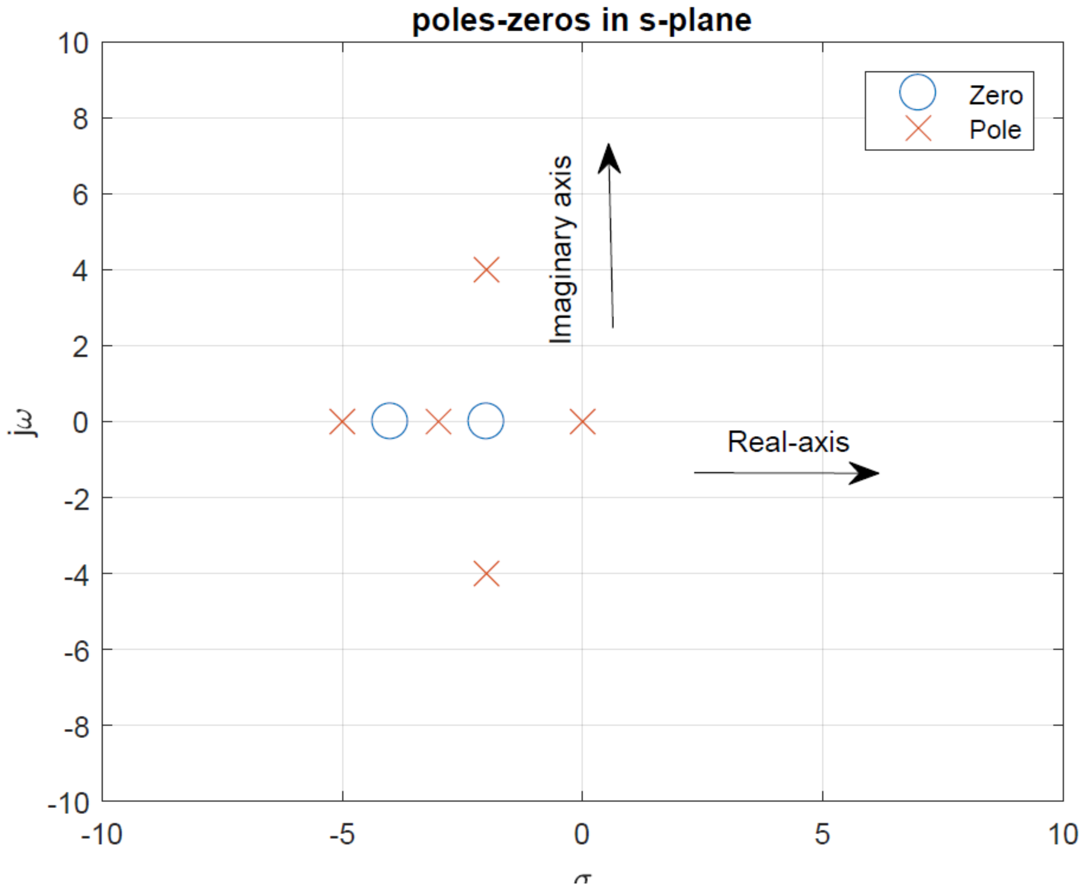


Fig 2.5. poles-zeros in s-plane.

In the transfer function expression of a control system the highest power of s in the numerator A(s) is either equal or less than that of the denominator B(s). The transfer function of a system is completely specified in terms of its poles and zeros and the gain factor.

2.3 Frequency domain Analysis, Characteristic parameters-corner frequencies, resonant frequencies, peak gain, band width-significance, Bode Plot, Polar plot, Nyquist plot, Experimental determination of Transfer function.

Frequency response: By the frequency response, we mean the steady state response of a system to a sinusoidal input. In frequency response method, we vary the frequency of input signal over certain range and study the resulting response. One advantage of the frequency response approach is that we can use the data obtained from the measurement on the physical system without deriving its mathematical model.

2.3.1 Frequency response of an open loop system: Consider the stable, linear time-invariant system shown below in fig 2.6.

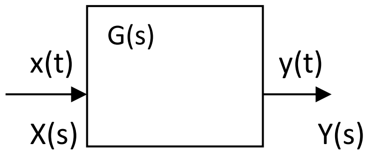


Fig 2.6: Open loop system

Y(s) = X(s) G(s)

Let x (t) = X sinωt)

Yss (t) = Ysin (ωt+ϕ); where Yss (t) is steady state out-put; Y = X$|G(j\omega)|$

And ϕ = $\lfloor G(j\omega)$ = tan $^{-1}\left[\frac{Imaginary\ \ part\ of\ G(j\omega)}{real\ part\ of\ G(j\omega)}\right]$

Φ = Phase shift; Y = gain.

A stable linear, time invariant system subject to a sinusoidal input will, at steady state, have a sinusoidal output of the same frequency as the input. But the amplitude and phase output will in general, be different from those of input. Amplitude of the output is given by the product of that of input and $|G(j\omega)|$, while the phase angle differs from that of input by the amount ϕ = $\lfloor G(j\omega$. The function G(jω) is called the **sinusoidal transfer function**. The sinusoidal transfer function is obtained by substituting jω for s in the transfer function of the system.

Example: Consider the system where transfer function G(s) is given by

G(s) = $\frac{K}{Ts+1}$.

For the sinusoidal input x(t) = X sin(ωt), the steady state output can be found as follows: substituting jω for s in G(s) yield

G(jω) = $\frac{K}{Tj\omega+1}$

The amplitude ratio of output to the input is

$|G(j\omega)|$ = K/$\sqrt{1+(\omega T)^2}$

While the phase angle is ϕ = -tan $^{-1}$ (Tω)

Hence Yss = XK/($\sqrt{1+(\omega T)^2}$) * Sin (ωt - tan $^{-1}$ Tω)

2.3.3 Frequency Response of a Closed Loop System:

The closed loop transfer function is given by the equation

M(s)= Y(s)/R(s) $= \frac{G(s)}{1+G(s)H(s)}$

Under the sinusoidal steady state, s= j ω

M(j ω) = Y(j ω)/R(j ω) $= \frac{Gj\omega()}{1+G(j\omega)H(j\omega)}$

Thee sinusoidal steady state transfer function M($j\omega$) may be expressed in terms of its magnitude and phase, that is,

M($j\omega$) = $|M(j\omega)|$ ∟M($j\omega$)

2.3.4 Frequency Domain Specifications (Characteristic parameters):

In the design of linear system using frequency domain methods, it is necessary to define a set of specifications so that the performance of the system can be identified. The frequency domain specifications are often used. They are as follows:

(a) **Resonant Peak Mr**: the resonant peak Mr is the maximum value of magnitude of M(jω). In general Mr gives indication of the relative stability of a stable closed loop system. Normally a large Mr corresponds to a large maximum overshoot of the step response. For most control systems M_r should be between 1.1 to 1.5. M_r is shown in fig 276 below.

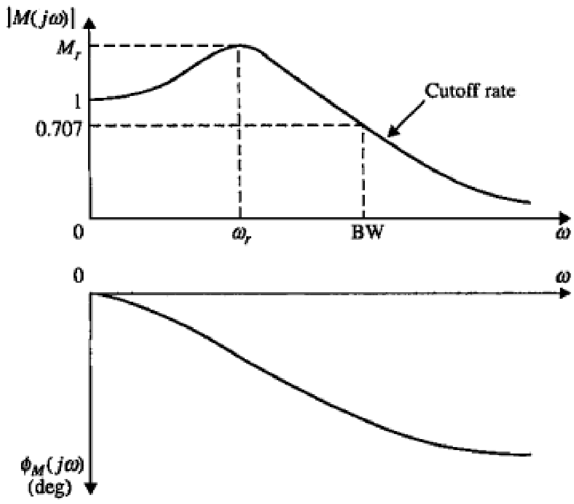


Fig 2.7: Frequency response characteristics.

For a second order system value of Mr is given by the following relation:

Mr $= \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ for $\zeta \leq .707$

It is important to note that for second order system, resonant peak depends only on damping ratio. For damping ratio greater than .707, no resonance occurs and resonant frequency is zero.

(b**) Resonant Frequency ω_r**: It is frequency at which the peak resonance Mr occurs. For a second order prototype system, resonant frequency is given by

$$\boldsymbol{\omega_r = \omega_n\sqrt{1-2\zeta^2}}$$

(c) **Bandwidth (BW):** The BW is the frequency at which magnitude of M (jω) drops to 70.7% of, or 3dB down from, its zero-frequency value. In general, the BW of a control system gives an indication of the transient response properties in time domain. A large BW corresponds to a faster rise time, since higher frequencies are more easily passed through the system. BW also indicates the noise-filtering characteristic & the robustness of the system. The response represents a measure of sensitivity of a system to parameter variations. A robust system is one i.e. insensitivity to parameter variations. BW for a second order system is given by the following formula

$$\mathbf{BW=}\left[(1-2\zeta^2)+\sqrt{4\,\zeta^4-4\,\zeta^2+2}\right]^{1/2}$$

Summary of relation of system parameters with characteristic parameters:

(a) Bandwidth and rise time are inversely proportional.

(b) Therefore, the larger the bandwidth is, the faster the system will respond.

(c) Increasing ω_n increases BW and decreases rise time t_r.

(d) Increasing ζ decreases BW and increases t_r.

2.4 Bode Plots. The Bode plot consists of two graphs. One is plot of the logarithmic of the magnitude of a sinusoidal transfer function, the other is a plot of the phase angle; both are plotted against the frequency on a logarithmic scale. The standard representation of the logarithmic magnitude of G(jω) is 20 log $|G(j\omega)|$, where the base of the logarithm is 10. The unit used is decibel, usually denoted as dB. In the logarithmic representation, the curves are drawn as semi log paper, using the log scale for frequency & linear scale for either magnitude (in decibel) or phase angle (in degrees). The main advantages of using the Bode diagram is that multiplication of magnitude can be converted into addition. Furthermore, a simple method of sketching an approximate log-magnitude curve is available. It is based on asymptotic approximations. Such approximations by straight line asymptotes are sufficient if only rough information on the frequency response characteristic is needed. Should the exact curve be desired, corrections can be made easily to these basic asymptotic plots.

Consider a control system with the following transfer function written in terms of poles and zeros

G(s)= $\frac{K(s+z1)(s+z2)\ldots(s+zm)}{s^j(s+p1)(s+p2)\ldots(s+pn)}$

This can also be written as

G(s)= $\frac{K1(1+T1s)(1+T2s)\ldots(1+Tms)}{s^j(1+Tas)(1+Tbs)\ldots(1+Tns)}$

The magnitude of G($j\omega$) in DB is obtained by multiplying the logarithm (base 10) of |G($j\omega$)| by 20; we have

20 $\log_{10}|G(j\omega)|$ =20 $\log_{10}|K|$ + 20$\log_{10}|(1+j\omega T1)|$ +20 $\log_{10}|1+j\omega T2|$-20$\log_{10}|j\omega|$

-20 $\log_{10}|1+j\omega Ta|$-20 $\log_{10}|1+j\omega Tb|$ -20 $\log_{10}\left|1+j2\zeta\omega-{\omega^2}/{\omega_n{}^2})\right|$ (2.8)

The phase of G($j\omega$) is

$\angle$G($j\omega$) = $\angle K$ + $\angle(1+j\omega T1)$ + $\angle(1+j\omega T2)$- $\angle$ $j\omega$ -$\angle(1+j\omega Ta)$- $\angle(1+j2\zeta\omega-{\omega^2}/{\omega_n{}^2})$

(2.9)

The Bode plot is a graph obtained from equation (2.8) and (2.9) consisting of two parts as follows:

(i) Magnitude of $G(j\omega)$ in decibel, {i.e. 20 $\log_{10}|G(j\omega)|$} versus $\log_{10}\omega$.

(ii) phase angle $\angle$G($j\omega$) =versus $\log_{10}\omega$.

For plotting magnitude versus $\log_{10}\omega$ each term in (2.8) is considered separately and graphs are drawn. To obtain final plot the contribution due to each term are added separately.

Example: as an illustrative example consider the Bode plot of the function

G(s)= $\frac{10(s+10)}{s(s+2)(s+5)}$

The first step is to express G(s) in the form of G($j\omega$) by replacing s with $j\omega$. We have

G($j\omega$) = $\frac{10(1+j0.1\omega)}{j\omega(1+j05.\omega(1+j0.2\omega)}$

Magnitude and phase plots (Bode plot) are shown in fig 2.8 below.

Bode plot can be obtained easily using MATLAB. Following is the MATLAB code for the same

```
% Bode Plot

% Transfer function G(s)=10*(s+10)/(s*(s+2)*(s+5))

s=tf('s');

Gs=10*s/(s*(s+2)*(s+5));    % Gs is the transfer function

bode(Gs);

grid on;
```

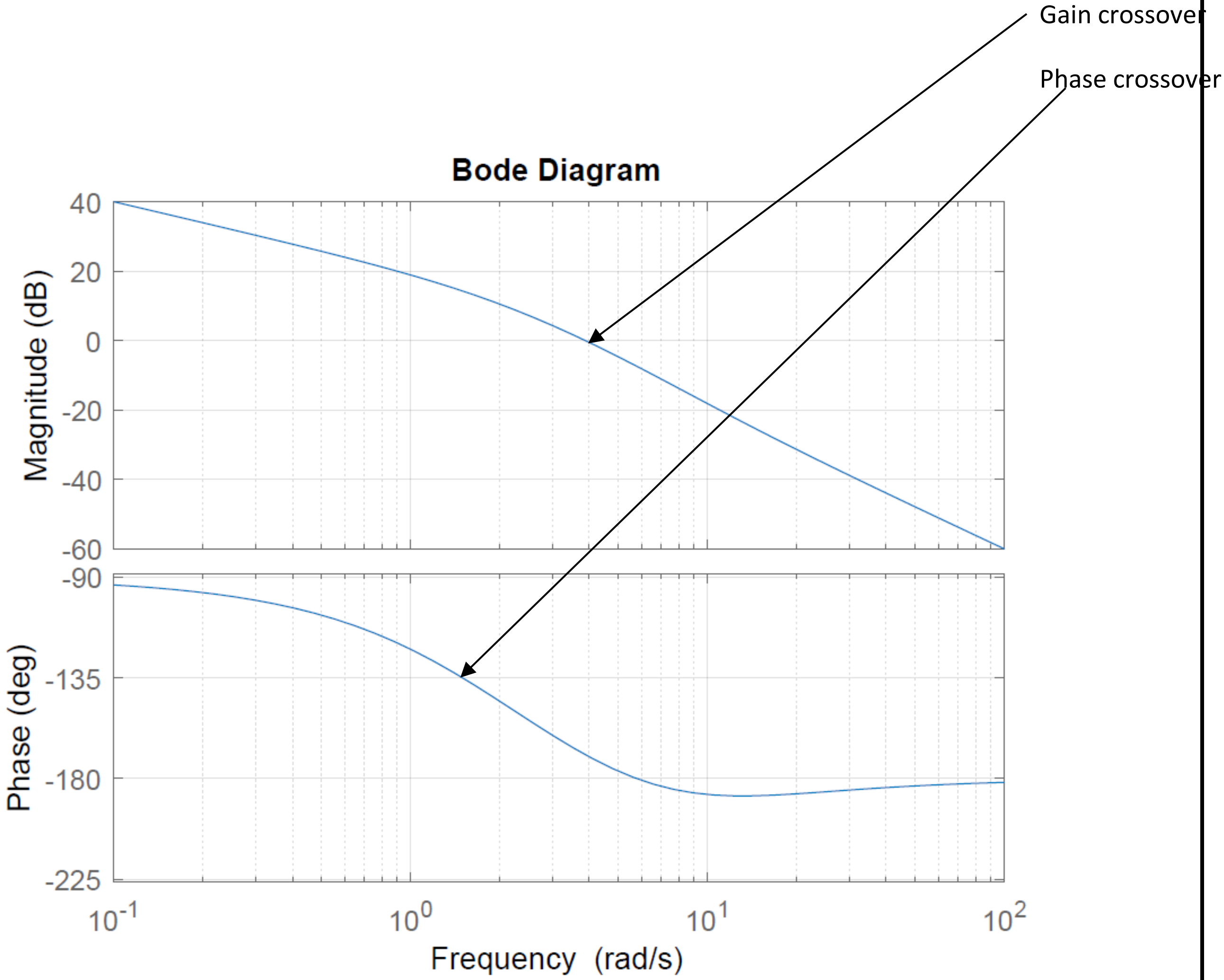


Figure 2.8 Bode plot of G(s)=10(s+10)/(s(s+2) (s+5))

Gain & Phase cross Over Points: gain and phase cross over points on frequency-domain plots are important for analysis and design of control systems.

Gain cross Over Point: the gain cross over point on the frequency plot of G(jω) is a point at which magnitude of G(jω) is unity. The frequency at the gain cross over point is called gain cross over frequency. This is shown in the figure 2.8 above.

Phase Cross Over Point: the phase cross over point on the frequency response curve of G(jω is a point at which phase angle of G(jω) = 180 degree. The frequency at the cross over point is called phase cross over frequency. This is shown in the figure 2.8 above.

Bode plot of a second order transfer function is given as

$$G(s)= \frac{\omega_n{}^2}{s^2+2\zeta\omega_n\ s+\omega_n{}^2}$$

The bode plot will depend on damping ratio zeta.

We can plot the Bode plot for different values of damping factor ζ. The plot is shown in figure 2.9

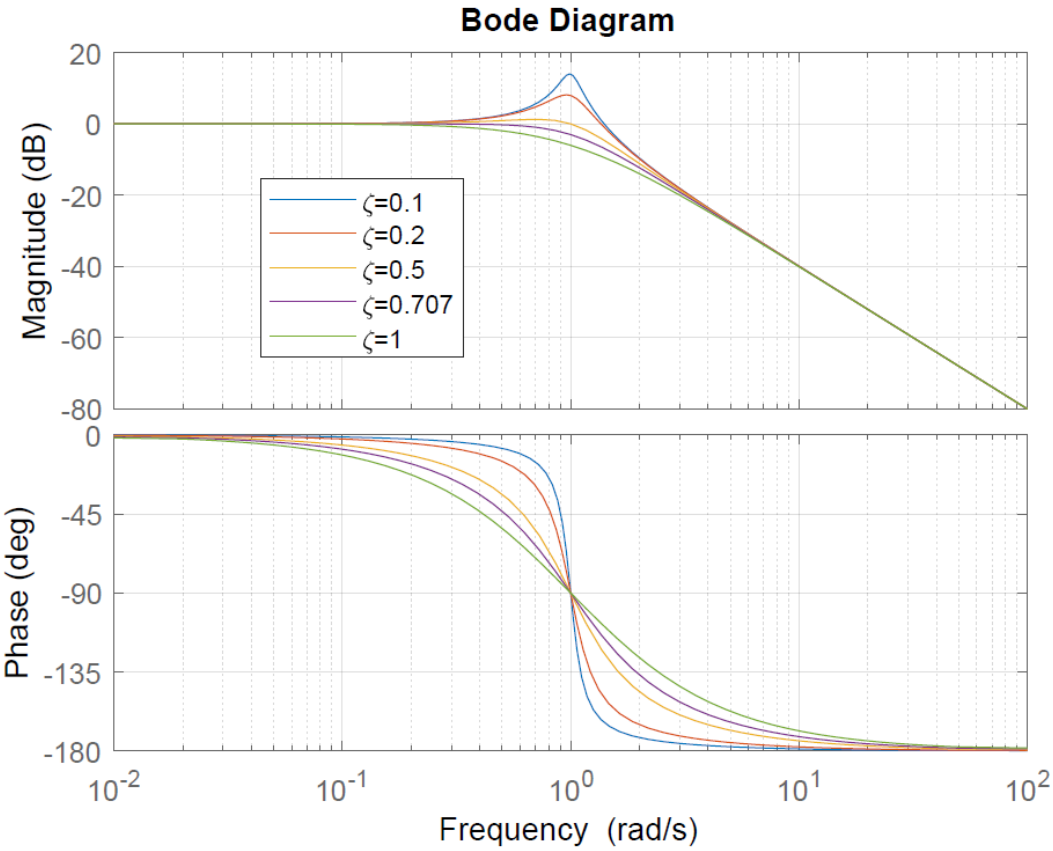


Figure 2.9. Bode plot of a second order system

2.4 Polar Plot (Nyquist Plot: The Polar plot of a sinusoidal transfer function G(jω) is a plot of the magnitude of G(jω) versus the phase of G(jω) on polar co-ordinates as ω is varied from zero to infinity. Note that in polar plots a positive (negative) phase angle is measured counterclockwise (clockwise) from the positive real axis. The polar plot is also called Nyquist plot.

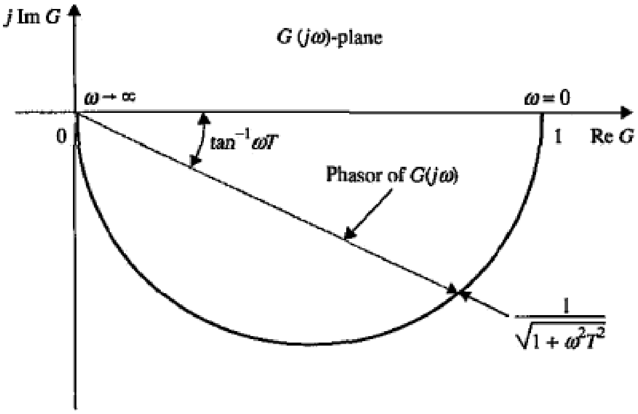


Figure 2-9 Polar plot of $G(j\omega) = \frac{1}{(1+j\omega T)}$.

On polar plot x-axis corresponds to real part of G(jω) and y-axis corresponds to Imaginary part of G(jω) as ω is varied from 0 to infinity. The polar plotof a function G(s)= 1/(1+Ts) is shown in the above figure.when ω is zero, the magnitude of G(jω) is unity, and the phase of G(jω) is at 0 deg. As ω is increased, the magnitude decreases, and the phase becomes more negative. As ω increases, the length of the vector in the polar coordinate decreases and the vector rotates in the clockwise(negative) direction. When ω approaches infinity, the magnitude G(jω) becomes zero, and the phase angle reaches -90 deg. This is presented by a vector with infinitesimally small length directed along the -90 deg-axis in the G(jω)-plane. By substituting other values of ω in the equation of G(s), the exact plot of G(jω) turns out to be a semicircle as shown in fig 2.9 above.

2.7 Experimental determination of system transfer functions by frequency response measurements. At times real world systems can be difficult to model mathematically. Fortunately, there is a convenient frequency response approach that allows experimental determination of the system transfer function. Frequency response can be determined by inputting a sinusoidal input at a varying frequency into a system. The output magnitude and frequency, which will also be sinusoidal, are then measured. The relationship between the input and output sinusoid at each sinusoidal frequency is then compared to produce a magnitude and phase at each frequency. A Bode plot can be constructed. Fig 2.10 below shows a block diagram of this equipment setup.

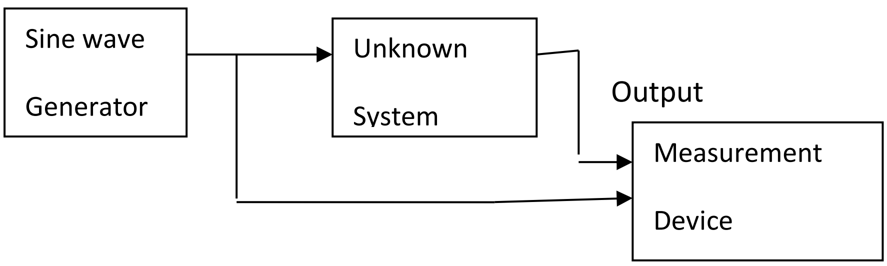


Fig 2.10: Experimental Frequency Response Setup.

A minimum phase system has no poles or zeros in the right-half s-plane, while a non-minimum phase system has at least one pole or zero in the right half s-plane. This affects the phase of a system. If a system is known to be minimum phase, the system transfer function can be obtained from the magnitude plot alone. If it is not known in advance, then both the magnitude & phase information are needed. After the Bode plot has been made from the set of experimental magnitude and phase data at different frequencies, the system transfer function can be obtained. The problem is to fit asymptotic approximation lines at the corner frequencies to determine pole/zero locations. The general procedure for finding the system transfer function given an experimental frequency response is as follows:

1. Find all single poles (-20 dB/decade changes).

2. Find all single zeroes (+20 dB/decade changes)

3. Find all double real poles (-40 dB/decade changes with no resonant peak.)

4. Find all double real zeros (+ 40 dB/decade changes with no resonant peak).

5. Find complex pole pairs (-40 dB/decade changes with a resonant peak)

6. Find complex zeroes pairs (+40 dB /decade changes with a resonant undershoot).

7. Find values for the Bode gain K (K = 1 or 0 dB before any pole/zero); if differentiator or integrator are present, look at the $\omega = 1$ point where both have values of 0 dB.

Appendix 'A': Laplace transform and Inverse Laplace Transform Table.

Appendix 'B': Theory of Complex Variable.

Appendix 'A': Laplace Transform Table

Laplace Transform Table

Laplace Transform $F(s)$	Time Function $f(t)$
1	Unit-impulse function $\delta(t)$
$\frac{1}{s}$	Unit-step function $u_s(t)$
$\frac{1}{s^2}$	Unit-ramp function t
$\frac{n!}{s^{n+1}}$	$t^n (n = \text{positive integer})$
$\frac{1}{s+\alpha}$	$e^{-\alpha t}$
$\frac{1}{(s+\alpha)^2}$	$te^{-\alpha t}$
$\frac{n!}{(s+\alpha)^{n+1}}$	$t^n e^{-\alpha t} (n = \text{positive integer})$
$\frac{1}{(s+\alpha)(s+\beta)}$	$\frac{1}{\beta-\alpha}(e^{-\alpha t} - e^{-\beta t})(\alpha \neq \beta)$
$\frac{s}{(s+\alpha)(s+\beta)}$	$\frac{1}{\beta-\alpha}(\beta e^{-\beta t} - \alpha e^{-\alpha t})(\alpha \neq \beta)$
$\frac{1}{s(s+\alpha)}$	$\frac{1}{\alpha}(1 - e^{-\alpha t})$
$\frac{1}{s(s+\alpha)^2}$	$\frac{1}{\alpha^2}(1 - e^{-\alpha t} - \alpha t e^{-\alpha t})$
$\frac{1}{s^2(s+\alpha)}$	$\frac{1}{\alpha^2}(\alpha t - 1 + e^{-\alpha t})$
$\frac{1}{s^2(s+\alpha)^2}$	$\frac{1}{\alpha^2}\left[t - \frac{2}{\alpha} + \left(t + \frac{2}{\alpha}\right)e^{-\alpha t}\right]$
$\frac{s}{(s+\alpha)^2}$	$(1 - \alpha t)e^{-\alpha t}$
$\frac{\omega_n}{s^2+\omega_n^2}$	$\sin \omega_n t$
$\frac{s}{s^2+\omega_n^2}$	$\cos \omega_n t$

Laplace Transform $F(s)$	Time Function $f(t)$
$\dfrac{\omega_n^2}{s(s^2+\omega_n^2)}$	$1-\cos\omega_n t$
$\dfrac{\omega_n^2(s+\alpha)}{s^2+\omega_n^2}$	$\omega_n\sqrt{\alpha^2+\omega_n^2}\,\sin(\omega_n t+\theta)$ where $\theta=\tan^{-1}(\omega_n/\alpha)$
$\dfrac{\omega_n}{(s+\alpha)(s^2+\omega_n^2)}$	$\dfrac{\omega_n}{\alpha^2+\omega_n^2}e^{-\alpha t}+\dfrac{1}{\sqrt{\alpha^2+\omega_n^2}}\sin(\omega_n t-\theta)$ where $\theta=\tan^{-1}(\omega_n/\alpha)$
$\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\dfrac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t \quad (\zeta<1)$
$\dfrac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$	$1-\dfrac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\theta\right)$ where $\theta=\cos^{-1}\zeta \quad (\zeta<1)$
$\dfrac{s\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\dfrac{-\omega_n^2}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t-\theta\right)$ where $\theta=\cos^{-1}\zeta \quad (\zeta<1)$
$\dfrac{\omega_n^2(s+\alpha)}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\omega_n\sqrt{\dfrac{\alpha^2-2\alpha\zeta\omega_n+\omega_n^2}{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\theta\right)$ where $\theta=\tan^{-1}\dfrac{\omega_n\sqrt{1-\zeta^2}}{\alpha-\zeta\omega_n} \quad (\zeta<1)$
$\dfrac{\omega_n^2}{s^2(s^2+2\zeta\omega_n s+\omega_n^2)}$	$t-\dfrac{2\zeta}{\omega_n}+\dfrac{1}{\omega_n\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\theta\right)$ where $\theta=\cos^{-1}(2\zeta^2-1) \quad (\zeta<1)$

Appendix 'B': Complex Variable Concept

2-1 COMPLEX-VARIABLE CONCEPT

To understand complex variables, it is wise to start with the concept of complex numbers and their mathematical properties.

2-1-1 Complex Numbers

A complex number is represented in **rectangular form** as

$$z = x + jy \tag{2-1}$$

where, $j = \sqrt{-1}$ and (x, y) are real and imaginary coefficients of z respectively. We can treat (x, y) as a point in the **Cartesian** coordinate frame shown in Fig. 2-1. A point in a

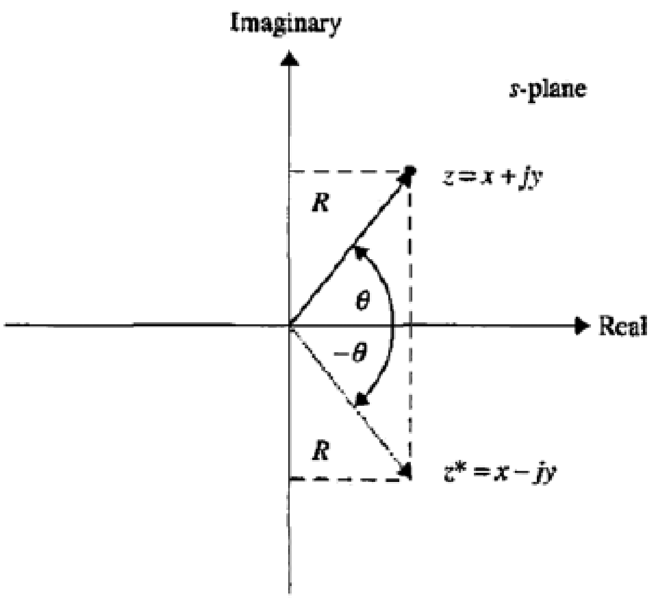


Figure 2-1 Complex number z representation in rectangular and polar forms.

rectangular coordinate frame may also be defined by a vector R and an angle θ. It is then easy to see that

$$x = R\cos\theta$$
$$y = R\sin\theta \tag{2-2}$$

where,

$R =$ magnitude of z

$\theta =$ phase of z and is measured from the x axis. Right-hand rule convention: positive phase is in counter clockwise direction.

Hence,

$$R = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\frac{y}{x} \tag{2-3}$$

Introducing Eq. (2-2) into Eq. (2-1), we get

$$z = R\cos\theta + jR\sin\theta \tag{2-4}$$

Upon comparison of Taylor series of the terms involved, it is easy to confirm

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{2-5}$$

Eq. (2-5) is also known as the **Euler formula.** As a result, Eq. (2-1) may also be represented in **polar form** as

$$z = Re^{j\theta} = R\angle\theta \tag{2-6}$$

We define the **conjugate** of the complex number z in Eq. (2-1) as

$$z^* = x - jy \tag{2-7}$$

Or, alternatively,

$$z^* = R\cos\theta - jR\sin\theta = Re^{-j\theta} \tag{2-8}$$

Note:

$$zz^* = R^2 = x^2 + y^2 \tag{2-9}$$

Table 2-1 shows basic mathematical properties of complex numbers.

TABLE 2-1 Basic Properties of Complex Numbers

Addition	$\begin{cases} z_1 = x_1 + jy_1 \\ z_2 = x_2 + jy_2 \end{cases}$ $\rightarrow z = (x_1 + x_2) + j(y_1 + y_2)$	$\begin{cases} z_1 = R_1 e^{j\theta_1} \\ z_2 = R_2 e^{j\theta_2} \end{cases}$ $\rightarrow z = R_1 e^{j\theta_1} + R_2 e^{j\theta_2}$
Subtraction	$\begin{cases} z_1 = x_1 + jy_1 \\ z_2 = x_2 + jy_2 \end{cases}$ $\rightarrow z = (x_1 - x_2) + j(y_1 - y_2)$	$\begin{cases} z_1 = R_1 e^{j\theta_1} \\ z_2 = R_2 e^{j\theta_2} \end{cases}$ $\rightarrow z = R_1 e^{j\theta_1} - R_2 e^{j\theta_2}$
Multiplication	$\begin{cases} z_1 = x_1 + jy_1 \\ z_2 = x_2 + jy_2 \end{cases}$ $\rightarrow z = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$ $j^2 = -1$	$\begin{cases} z_1 = R_1 e^{j\theta_1} \\ z_2 = R_2 e^{j\theta_2} \end{cases}$ $\rightarrow z = (R_1 R_2) e^{j(\theta_1 + \theta_2)}$ $\rightarrow z = (R_1 R_2) \angle(\theta_1 + \theta_2)$
Division	$\begin{cases} z_1 = x_1 + jy_1 \\ z_2 = x_2 + jy_2 \end{cases}$ $\begin{cases} z_1^* = x_1 - jy_1 \\ z_2^* = x_2 - jy_2 \end{cases}$ *Complex Conjugate* $\rightarrow z = \dfrac{z_1}{z_2}$ $\rightarrow z = \dfrac{z_1 z_2^*}{z_2 z_2^*} = \dfrac{(x_1 x_2 + y_1 y_2) + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$	$\begin{cases} z_1 = R_1 e^{j\theta_1} \\ z_2 = R_2 e^{j\theta_2} \end{cases}$ $\begin{cases} z_1^* = R_1 e^{-j\theta_1} \\ z_2^* = R_2 e^{-j\theta_2} \end{cases}$ $\rightarrow z = \left(\dfrac{R_1}{R_2}\right) e^{j(\theta_1 - \theta_2)}$ $\rightarrow z = \left(\dfrac{R_1}{R_2}\right) \angle(\theta_1 - \theta_2)$

2-1-2 Complex Variables

A complex variable s has two components: a real component σ and an imaginary component ω. Graphically, *the real component of* s *is represented by a* σ *axis* in the horizontal direction, and *the imaginary component is measured along the vertical* $j\omega$ *axis*, in the complex s-plane. Fig. 2-2 illustrates the complex s-plane, in which any arbitrary point $s = s_1$ is defined by the coordinates $\sigma = \sigma_1$, and $\omega = \omega_1$, or simply $s_1 = \sigma_1 + j\omega_1$.

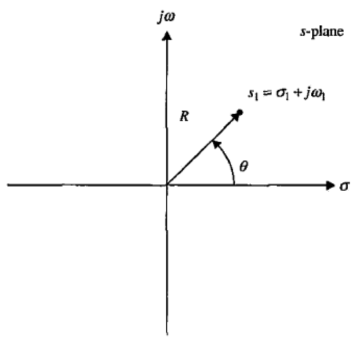


Figure 2-2 Complex s-plane.

2-1-3 Functions of a Complex Variable

The function $G(s)$ is said to be a function of the complex variable s if, for every value of s, there is one or more corresponding values of $G(s)$. Because s is defined to have real and imaginary parts, the function $G(s)$ is also represented by its real and imaginary parts; that is,

$$G(s) = \mathrm{Re}[G(s)] + j\,\mathrm{Im}[G(s)] \tag{2-11}$$

where $\mathrm{Re}[G(s)]$ denotes the real part of $G(s)$, and $\mathrm{Im}[G(s)]$ represents the imaginary part of $G(s)$. The function $G(s)$ is also represented by the complex $G(s)$-plane, with $\mathrm{Re}[G(s)]$ as the real axis and $\mathrm{Im}[G(s)]$ as the imaginary axis. If for every value of s there is only one corresponding value of $G(s)$ in the $G(s)$-plane, $G(s)$ is said to be a **single-valued function**, and the mapping from points in the s-plane onto points in the $G(s)$-plane is described as **single-valued** (Fig. 2-3). If the mapping from the $G(s)$-plane to the s-plane is also single-valued, the mapping is called **one-to-one**. However, there are many functions for which the mapping from the function plane to the complex-variable plane is not single-valued. For instance, given the function

$$G(s) = \frac{1}{s(s+1)} \tag{2-12}$$

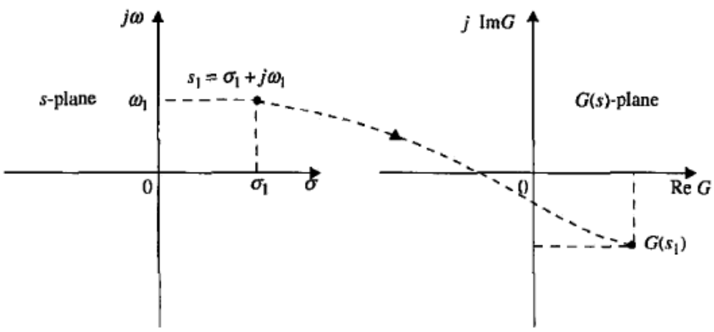


Figure 2-3 Single-valued mapping from the s-plane to the $G(s)$-plane.

2-1-4 Analytic Function

A function G(s) *of the complex variable* s *is called an* ***analytic function*** *in a region of the* s-*plane if the function and all its derivatives exist in the region.* For instance, the function given in Eq. (2-12) is analytic at every point in the s-plane except at the points $s = 0$ and $s = -1$. At these two points, the value of the function is infinite. As another example, the function $G(s) = s + 2$ is analytic at every point in the finite s-plane.

The **singularities** of a function are the points in the s-plane at which the function or its derivatives do not exist. A pole is the most common type of singularity and plays a very important role in the studies of classical control theory.

UNIT-III

DESIGN OF CONTROL SYSTEMS.

Index

3. Introduction: A controlled system is shown in the following block diagram (fig 3.1)

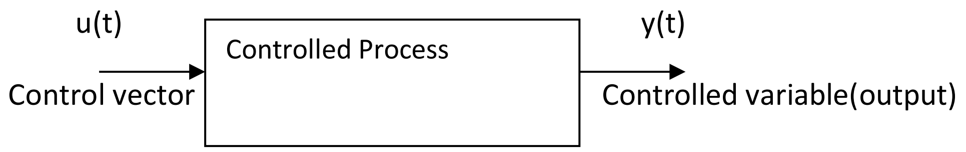


Fig 3.1 Controlled Processes

Control system specifications and design involves the following steps.

(a) Determine what the system should do and how to do it (Performance and design specification).

(b) Determine the controller or compensator configuration relative to how it is connected to the controlled process.

(c) Determine the parameter values of the controller to achieve the design.

3.1 Control System Performance requirements:

3.1.1 Transient and Steady state specifications.

When an input is applied to a control system, the output may be oscillatory for some time before reaching the final or steady state value. Steady state value is the output as time approaches infinity.

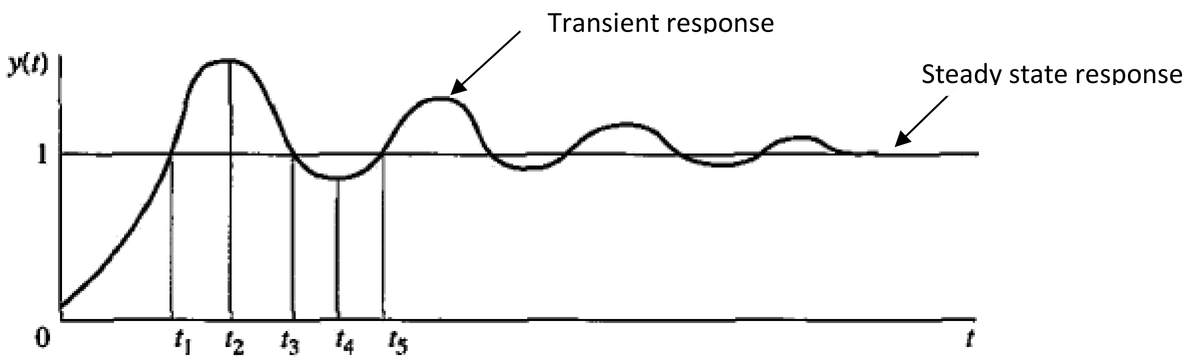


Fig 3.2 Transient and steady state response

Transient and steady state response of a control system with unit feedback is shown in the fig 3.2 above for a unit step input. Desired output is also a unit step function. As could be seen output y (t) reaches unity value as t approaches a large value. In the above control system, we would like that output follows the input accurately. Which means system should not have any error or 100% accurate. But real-world control problem seldom follows this because of various reasons like non-linearity, friction, aging of components etc. Transient response specifications can be specified in terms of rise time, delay time, settling time (speed of response), and percentage overshoot (Mp) etc. Steady error is the error between output and input as time approaches infinity. Robustness is the ability of the system to be insensitive to system parameter variations (like Gain etc) and to external disturbance and also to noise.

3.1.2. Relations with system parameters; Example of first and second order system: Performance specifications discussed above are related to system parameters as discussed in the following paragraphs:

(a) First order System. We know that a first order system is described by the following transfer function: G(s) = 1/(Ts+1); where T is called the time, constant and is system parameter. The response of the system depends upon the time constant T. Lower the value of T, faster is the response. The response of the system to impulse input is given below (fig 3.3):

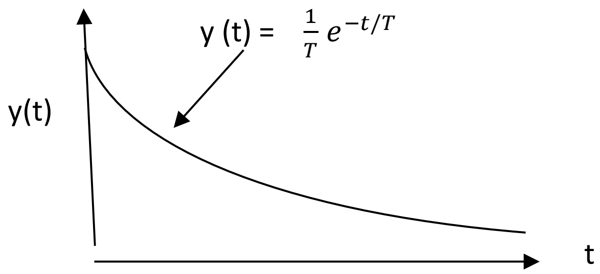


Fig: 3.3: Response of a first order system to impulse input

When the input is unit step function the out y (t) is given by the equation

y (t) = = 1- $e^{-t/T}$, for t >= 0. The response is given in the fig 3.4.

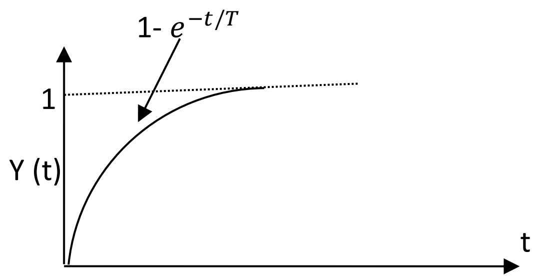


Fig 3.4 Response of first order system to step input

We find that for a step input steady state is zero for a first order system. Also we find that any change in system parameter T will affect the output, hence system is sensitive to parameter variation T due to aging of components which determine the value of T (e.g. Value of resistance and capacitance in a RC network).

(b) **Second order System**: A second order system is characterized by the following transfer function:

$C(s)/R(s) = \omega_n{}^2 / (s^2 + 2\zeta\, \omega_n\, s + \omega n^{\,2})$

ω_n is called un damped natural frequency, and ζ is called the damping ratio of the system. These are called **system parameters**. Speed of response of second order system is defined by rise time, delay time, settling time. Stability of the system depends on damping ratio a ζ. Similarly steady state error depends on type of input. For a step input steady state error is zero. Block diagram of a second order system is given in fig 3.5.

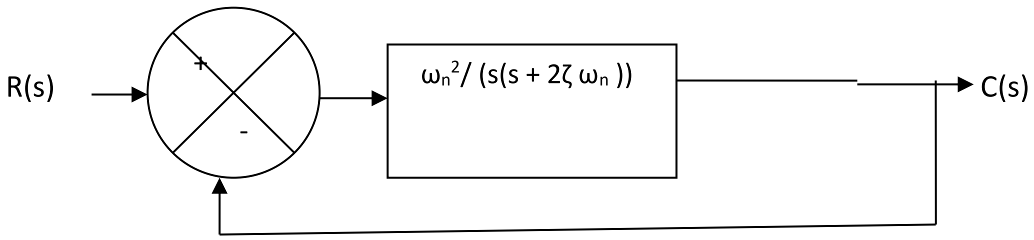


Fig: 3.5 Block diagram of second order System

3.1.3 Specifications in time domain, frequency domain, and 's' domain:

3.1.3.1 Specifications in time domain: Time domain specifications are as follows:

(i) Delay time, t_d.
(ii) Rise time, t_r.
(iii) Peak time, t_p.
(iv) Maximum overshoots, M_p.
(v)Settling time, t_s
(vi) Steady state error.
These specifications are shown in the Fig 3.6
Their relations with system parameters i.e. ω_n and ζ are as follows:

(i) Delay time t_d: The delay time is the time required for the response to half the final value for the first time. The delay time is related to system parameters for a second order system by:

$t_d \cong (1.\ 0.7\xi)/\omega n \quad ; 0< \xi < 1.0$

We can obtain a better approximation by using a second order equation

$t_d \cong (1.1 + 0.25\xi + 0.469\xi^{\,2})/\omega_n$

(ii) Rise time t_r: The rise time is the time required for the response to rise from 10% to90% of its final value.

Rise time $t_r \cong (0.8+2.5\zeta)/\omega_n$

(iii) Peak time Tp: The peak time Tp is the time required for the response to reach the first peak of the overshoot.

(iv) Maximum Overshoot, Mp: The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by:

Maximum percent overshoot = 100 *(c (t) − c (∞))/c (∞). The amount of maximum (percent) overshoot indicates the relative stability of the system.

% Maximum overshoot = 100 $e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$

(v) Settling time ts: The settling time ts is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).

Settling time for 5%

$t_s \cong 3.2/(\zeta\,\omega_n)$ for $0<\zeta<0.69$

$t_s \cong 4.5\zeta/\omega_n$; $\xi > 0.69$

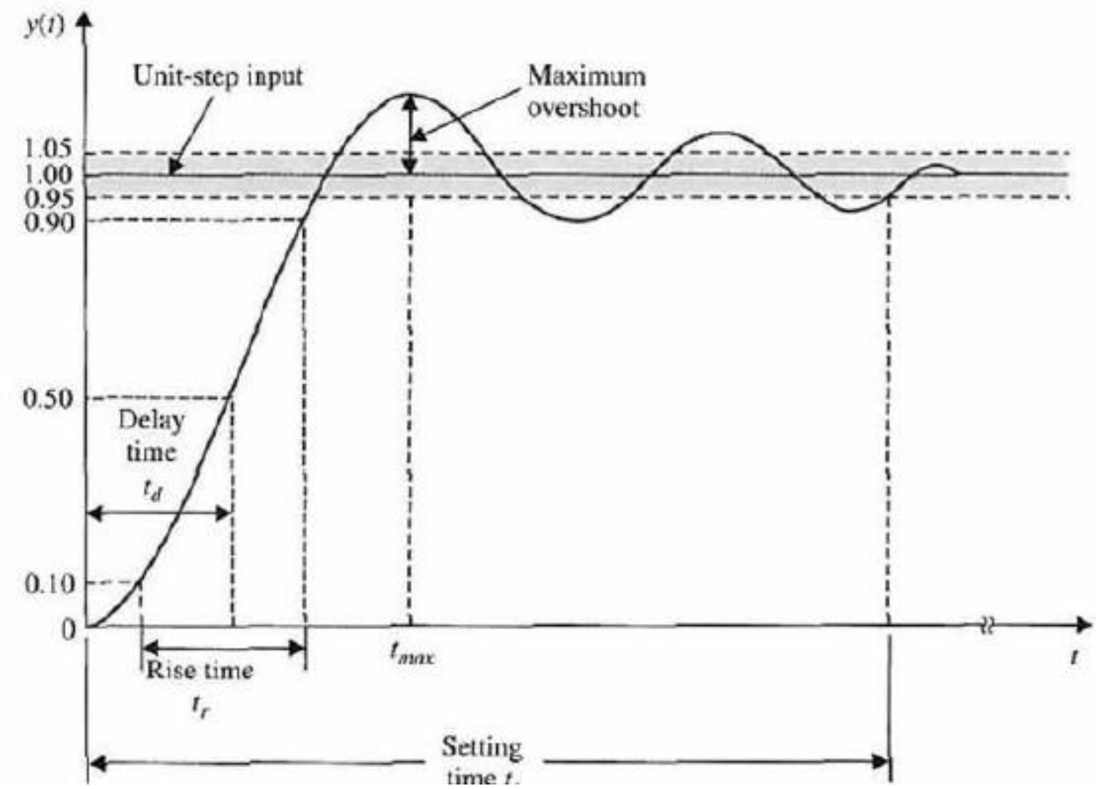


Fig 3.6: Second order system time domain specifications

3.1.3.2 Specifications in frequency domain: Frequency domain specifications are as follows:

(i) Resonant Peak Mr: the resonant peak Mr is the maximum value of magnitude of M (jω). In general Mr gives indication of the relative stability of a stable closed loop system. Normally a large Mr corresponds to a large maximum overshoot of the step response. For most control systems Mr should be between 1.1 to 1.5. Mr is shown in fig 2.4.7 below.

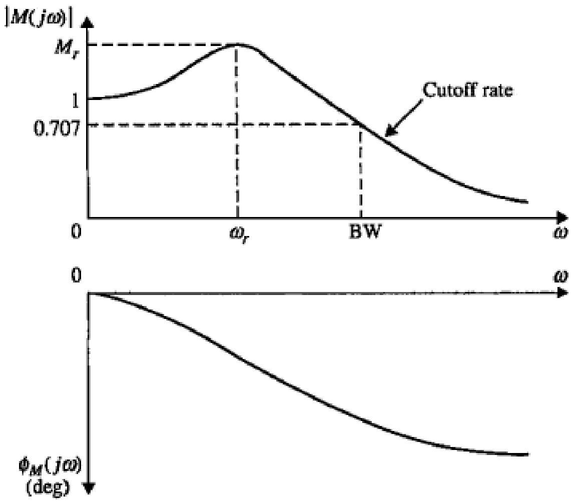


Fig 2.4.7: Frequency response characteristics.

(ii**) Resonant Frequency ω_r**: it is frequency at which the peak resonance Mr occurs.

(iii) **Bandwidth (BW):** The BW is the frequency at which magnitude of M (jω) drops to 70.7% of, or 3dB down from, its zero-frequency value. In general, the BW of a control system gives an indication of the transient response properties in time domain. A large BW corresponds to a faster rise time, since higher frequencies are more easily passed through the system. BW also indicates the noise-filtering characteristic & the robustness of the system. The response represents a measure of sensitivity of a system to parameter variations. A robust system is one i.e. insensitivity to parameter variations.

(iv) Gain Margin. It is a parameter which indicates the amount by which gain of a system can be increased before the system becomes unstable. It is specified in dB. A gain margin of <0 dB indicates instability. As a rule of thumb, we would like to have GM > 6 dB.

(v) Phase Margin: It is specified as an angle by which phase can be increased before the system becomes unstable. A phase margin < 0° indicates instability. As a rule of thumb, we would like to have 30°< PM < 60°.

3.1.3.3. Relation of Frequency Domain specifications with system parameters: Frequency domain specifications are related to system parameters by the following equations.

(i) **Resonant Frequency**: $\omega_r = \omega n\sqrt{1-2\zeta^2}$

(ii) **Resonant Peak**: Mr = $1/(2\zeta\sqrt{1-\zeta^2})$; for $\zeta \leq 0.707$

(iii) **Band width**= $\omega_n \left[(1-2\zeta^2)+\sqrt{2\zeta\zeta\zeta^2-4\zeta^2+2}\right]^{½}$

3.1.4.'s' domain specifications: Transfer function of a control system is function of 's' where s is a complex variable. Transfer function of a closed loop control system is given as:

TF = $\frac{G(s)}{1+G(s)H(s)}$; where G(s) is the transfer function of forward path element and H(s) is the transfer function of the element in the feedback path. We can use the characteristic equation in s to find the stability of the control system using Routh-Hurwitz criterion. Also for a given closed loop transfer function we can determine the poles and zeros which will affect the stability of a control system. Root-Locus method can be used to study the effect of system parameter variations on the stability of control system. As the poles move away from the origin of s-plane towards left half of imaginary axis system becomes more stable. Hence performance specifications can be specified in terms of position of poles and zeros. In general s-domain specifications provide following guide lines:

(i) Complex-conjugate poles of the closed- loop transfer function lead to a step response that is under damped. If all the system poles are real, the step response is over damped. However, zeros of the closed loop transfer function may cause overshoot even if the system is over damped.

(ii) The response of a system is dominated by those poles closest to the origin in the s-plane. Transients due to those poles farther to the left decay faster.

(iii) The farther to the left in the s-plane the system's dominant poles are, the faster the system will respond and the greater its bandwidth will be.

(iv) When a pole and zero of a system transfer function nearly cancel each other, the portion of the system response associated with the pole will have a small magnitude.

(v) Steady state error constants can be derived from the transfer function of the elements in the forward path and feedback path along with Laplace transform of input.

3.1.6 Method of determining Stability: Stability is of prime importance in control system. An unstable system is of no use. Let u (t), y(t), and g(t) be the input, output, and impulse response of a linear time-invariant system, respectively. With zero initial conditions, the system is said to be bounded-input bounded-output (BIBO) stable, or simply stable, if its output y (t) is bounded to a bounded input u(t). Roots of the characteristic equation determine the stability of a control system. If any of the roots lie on the right-half of the s-plane, system is unstable. Following methods are used for determining the stability of a control system, without involving root solving.

(a) Routh-Hurwitz criterion. This criterion is an algebraic method that provides information on the absolute stability of a linear time-invariant system that has a characteristic equation with constant coefficients. The criterion tests whether any of the roots of the characteristic equation lie in the right –half s-plane. A simple means of determining the absolute stability of a system can be obtained by Routh stability criterion. The method allows us to determine whether any roots of the characteristics equation have positive real parts, without actually solving the roots. Consider the characteristic equation

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots a_1 s^1 + a_0 \qquad (3.1)$$

So that no roots of equation (3.1) have positive real parts the necessary but not sufficient conditions are that

1. All the coefficients of the characteristic equation must have the same sign.

2. all the coefficients must exist.

To apply the Routh criterion, we must first define the Routh array as in table 3.1. The Routh array is continued horizontally and vertically until only zeros are obtained. The last step is to investigate the sign of the numbers in the first column of the Routh table. The Routh stability criterion states

1. If all the numbers of the first column have the same sign then the roots of the characteristic polynomial have negative real parts. The system is therefore stable.

2. If the numbers in the first column change sign then the number of sign changes indicates the number of roots of the characteristic equation having positive real parts. Therefore, if there is a sign change in the first column the system will be unstable.

Table 3.1 Definition of Routh-Hurwitz array: Routh table

---

s^n	a_n	a_{n-2}	a_{n-4}	…
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	…
s^{n-2}	b_1	b_2	b_3	
…	c_1	c_2	c_3	….

Where a_n, a_{n-1} ,…, a_0 are the coefficient of the characteristic equation and the coefficients b_1, b_2, b_3,c_1,c_2, and so on are given by

$b_1 \equiv \frac{a_{n-1}a_{n-2}-a_n a_{n-1}}{a_{n-1}}$; $b_2 \equiv \frac{a_{n-1}a_{n-4}-a_n a_{n-5}}{a_{n-1}}$ and so forth

$c_1 \equiv \frac{b_1 a_{n-3}-a_{n-1}b_2}{b_1}$; $c_2 \equiv \frac{b_1 a_{n-5}-a_{n-1}b_3}{b_1}$ and so forth

$d_1 \equiv \frac{c_1 b_2 - c_2 b_1}{c_1}$ and so forth

When developing the Routh array, several difficulties may occur. For example, the first number in one of the rows may be 0, but the other numbers in the row may not be. Obviously, if 0 appears in the first position of a row, the elements in the following row will be infinite. In this case, Routh test breaks down. Another possibility is that all the numbers in a row are zero. Methods of handling such cases will be illustrated with few examples.

Example 1. Determine whether the characteristic equation given below have stable or unstable roots.

(a) $s^3 + 6s^2 + 12s + 8 = 0$

(b) $s^3 + 4s^2 + 4s + 12 = 0$

(c) $As^4 + Bs^3 + Cs^2 + ds + E = 0$

Solution:

(a) The first two rows of the array are written down by inspection and the succeeding rows are obtained by using the relationship for each row element as presented previously:

$$\begin{matrix} 1 & 12 & 0 \\ 6 & 8 & 0 \end{matrix}$$

$$\frac{64}{6} \quad 0$$

$$8$$

There are no sign change in the column 1; therefore, the system is stable.

(b) The Routh array is as follows:

$$\begin{array}{ccc} 2 & 4 & 0 \\ 4 & 12 & 0 \end{array}$$

$$-2 \quad 0$$

$$12$$

Note that there are two sign changes in column 1; therefore, the characteristic equation has two roots with positive real parts. The system is unstable.

(c) The Routh stability criterion can be applied to the quartic characteristic equation that descrbes either the longitudinal or lateral motion of an airplane. The quartic characteristic equation for either the longitudinal or lateral equation of motion is given in terms of A,B,C,D which are functions of the longitudinal or lateral stability derivatives. Forming the Routh array from the characteristic equation yields

A	C	E
B	D	0
(BC-AD)/B	E	0
{[D(BC-AC)/B]-BE}/(BC-AD)/B	0	
E		

For the airplane to be stable requires that

A, B, C, D, E > 0;

BC-AC > 0

D(BC-AD)-B^2 E > 0

The last two inequalities were obtained by inspection of the first column of the Routh array.

If the first number in a row is 0 and the remaining elements of that row are nonzero, the Routh method breaks down. To overcome this problem the lead element that is 0 is replaced by a small positive number, ϵ . with the substitution of ϵ as the first element, the Routh array can be completed. After completing the Routh array, we can determine the first column to determine whether there is any sign change in the first column as ϵ approaches 0.

The other potential difficulty occurs when a complete row of the Routh array is 0. Again the Routh method breaks down. When this condition occurs, it means that that are symmetrically located roots in the s-plane. The roots may be real with opposite sign or complex conjugate roots. The polynomial formed by the coefficient of the first row just above the row of zeroes is called the auxiliary polynomial. The roots of the auxiliary polynomial are symmetrical roots of characteristic equation. The situation can be overcome by replacing the row of zeroes by the coefficients of the polynomial obtained by taking the derivative of the auxiliary polynomial. These exceptions are illustrated by way of few examples.

Example2. Determine the stability of the system represented by the following characteristic equations

(a) $s^5+s^4+3s^3+3s^2$+4s+6=0

(b) $s^6+3s^5+6s^4+12s^3+11s^2$+9s+6=0

Solution:

(a) For equation the element of the third row of the Routh table is 0 which prevents us from completing the table. This difficulty is avoided by replacing the lead element 0 in the third row by a small positive value ϵ . with the 0 removed and replaced by ϵ the Routh table can be completed as follows:

$$\begin{array}{ccc} 1 & 3 & 4 \\ 1 & 3 & 6 \\ \frac{3\epsilon+2}{\epsilon} & 6 & \\ \frac{-6\epsilon^2-6\epsilon-4}{3\epsilon+2} & 0 & \\ 6 & & \end{array}$$

Now as ϵ goes to 0 the sign of the first element in row 3 and 4 are positive. However, in row 5 the lead element goes to -2 as ϵ goes to 0. We note two sign changes in the first column of the Routh tables; therefore, the system has two roots with positive real parts, which means it is unstable. (b) the Routh table can be constructed as follows:

1	6	11	6
3	12	9	
2	8	6	
0	0		

Note that the fourth row of the Routh table is all zeros. The auxiliary equation is formed from the coefficients in the row just above the row of zeroes. For this example the auxiliary equation is

$2s^4+8s^2+6=0$

Taking the derivative of the auxiliary equation yields

$8s^3+16s=0$

The row of zeroes in the fourth row is replaced by the coefficients 8 and 16. The Routh table can now be completed.

1	6	11	6
3	12	9	
2	8	6	
8	16		
4	6		
4	0		
6			

The auxiliary equation can also be solved to determine the symmetric roots,

$s^4+4s^2+3=0$

Which can be factored as follows:

$(s^2+1)(s^2+3)=0$

Or s $=\pm i$ and s$=\pm\sqrt{3}i$

If we examine column 1 of the Routh table we conclude that there are no roots with positive real parts. However, solution of the auxiliary equations reveals that we have two pairs of complex roots lying on the imaginary axis. The purely imaginary roots lead to undamped oscillatory motions. In the absolute sense, the system is stable; that is, no part of the motion is growing with time. However, the purely oscillatory motions would be unacceptable for a control system.

(b) Nyquist criterion. This criterion is a semi graphical method that gives information on the difference between the number of poles and zeros of the closed loop transfer function that are in the right-half s-plane by observing the behavior of the Nyquist plot of the loop transfer function.

(c) Bode Diagram: This diagram is a plot of the magnitude of the loop transfer function G(jω)H(jω) in decibels and the phase of G(jω)H(jω) in degrees, all versus frequency ω. The stability of the closed loop system can be determined by observing the behavior of these plots.

3.2. Design of Controllers (Compensation Systems).

3.2.1 Need for Compensation: The goal of compensation is to augment the performance of the response so that it falls within desired specifications. These specifications can be in time domain (rise time; delay time, settling time, peak overshoot, steady state error (accuracy), relative stability) or in frequency domain (resonant frequency, band width, gain margin, phase margin, resonant peak). This can be done by placing a transfer function in various locations either inside or outside of the feedback loop. Figure 3.7 shows the three compensator locations-pre-filter, forward path (cascade), and feedback. In many control systems, the compensation device is an electrical circuit. Other forms of compensators may include mechanical, hydraulic, and pneumatic devices.

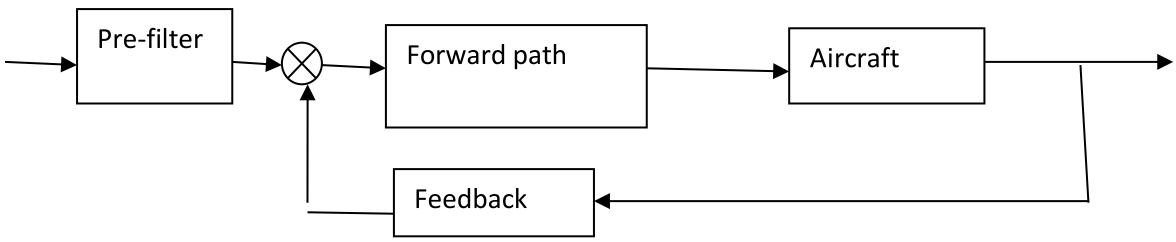


Fig 3.7 Compensator Locations

3.2.2 Active and passive Compensators: Passive compensators can be realized using resistor and capacitor devices. Low pass filter, high pass filter, differentiator; integrator can be realized using passive electrical circuits. Active compensation devices use operational amplifiers (OP-Amp). A high pass filter using passive compensator is shown above (Fig 3.8).

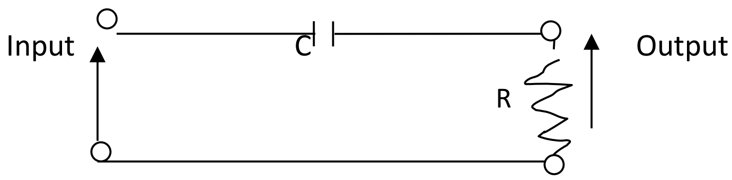


Fig 3.8: High Pass Filter using passive compensator

Same can be implemented using active compensator as shown below in fig 3.9.

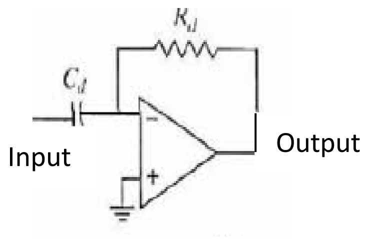


Fig 3.9: A high pass filter using active compensator (OP-Amp)

3.2.3 Series Compensation: A series compensator is shown in the fig 3.10 below.

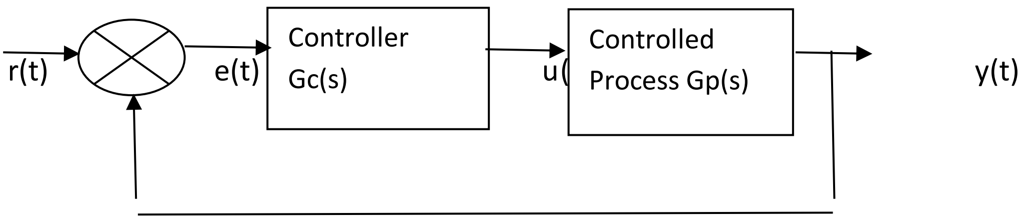


Fig 3.10: Series or Cascade Compensation

3.2.4. Feed Forward Compensation: Fig 3.11 and 3.12 show feed forward compensation. In fig 3.11, the feed forward controller G_{cf} (s) is placed in series with the closed –loop system, which has a controller G_c (s) in the forward path. In fig 3.12 the feed forward controller G_{cf} (s) is placed in parallel with the forward path.

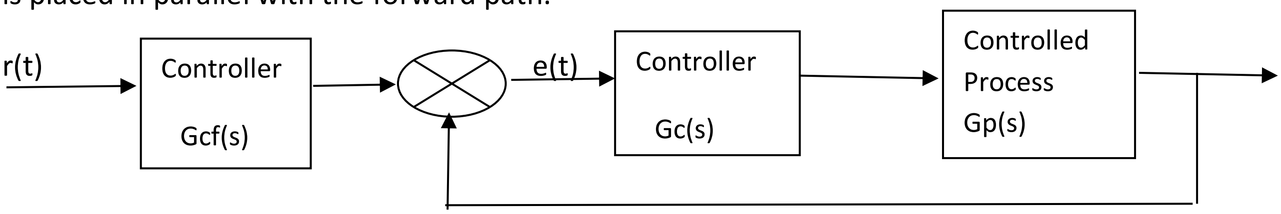


Fig 3.11: Feed forward compensation with series compensation (2-D freedom)

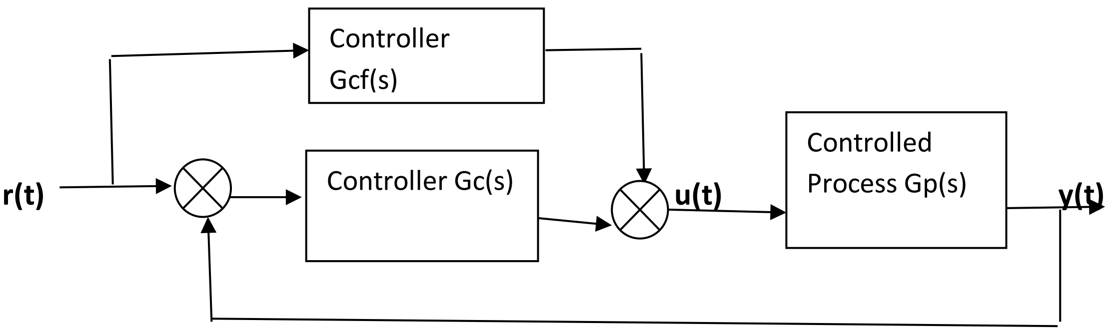


Fig 3.12 Feedforward Compensation (2-D of freedom)

3.2.5 Feedback Compensation: Feedback compensation is shown in fig 3.13 below.

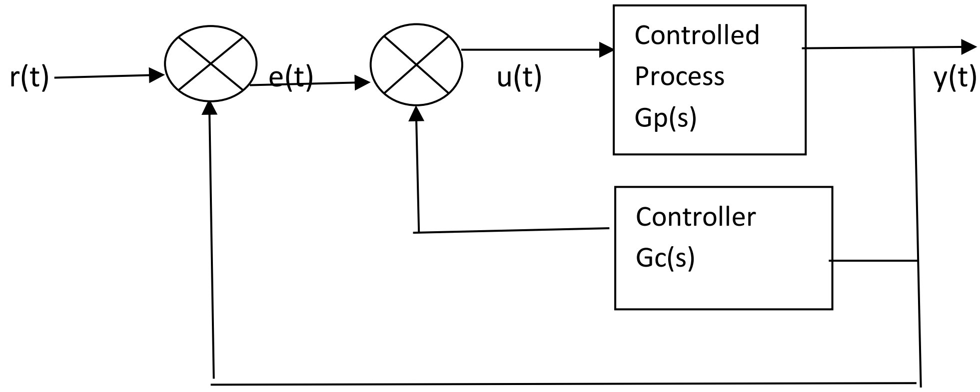


Fig 3.13: Feedback compensation

3.3.1 Proportional Control (**P Controller**): Fig 3.14 shows a second order prototype control system with proportional controller whose transfer function $G_c(s)$ = Kp; (where Kp is proportional gain).

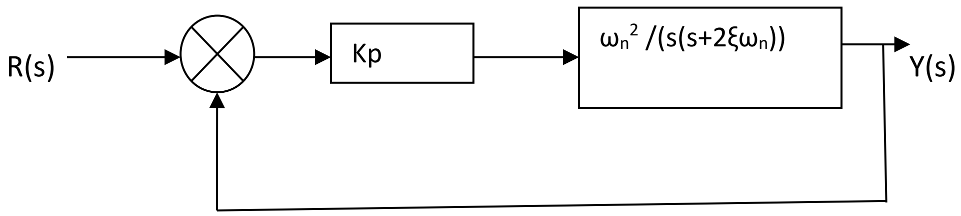


Fig 3.14: Proportional Controller

Closed loop transfer function Y(s)/R(s) is given as: $M(s) = Kp\ \omega_n^2 / (s^2 + 2\ \xi\ \omega_n s + Kp\ \omega_n^2)$. We can see that un- damped natural frequency ω_n has been increased to Kp $\sqrt{\omega n}$. Since rise time is inversely proportional to un- damped natural frequency, proportional controller reduces the rise time. Another merit of proportional controller is its simplicity. However, it increases the overshoot. Also, there may be steady state error.

3.3.2 Integral controller: An integral controller has transfer function as $G_c(s) = K_I /s$. (K_I is the integral gain). A block diagram of integral controller is given in fig 3.15.

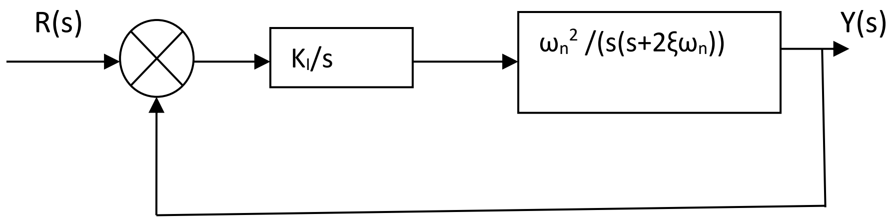


Fig 3.15: An integral controller

An integrator is an ideal **low-pass compensator**. It amplifies the low frequency (because Gc (ω) = KI/jω) while high frequencies are attenuated. The use of pure integral has the disadvantage of excessive lag. In addition, it has phase of -90° , which is a phase lag. This tends to **slow down the response**. An integral controller increases the system type by one; hence it **reduces the steady state error**. The disadvantage of the integral controller is that it makes the **system less stable** by adding the pole at the origin.

3.3.3 Proportional plus Derivative Control (PD Control): A more usable type of high-pass filter is proportional plus derivative (PD) high-pass filter. The Block diagram of a PD controller is shown in fig 3.16. The transfer function of PD controller is Gc(s) = K_p +s K_d; where K_p is proportional gain and K_d is derivative gain.

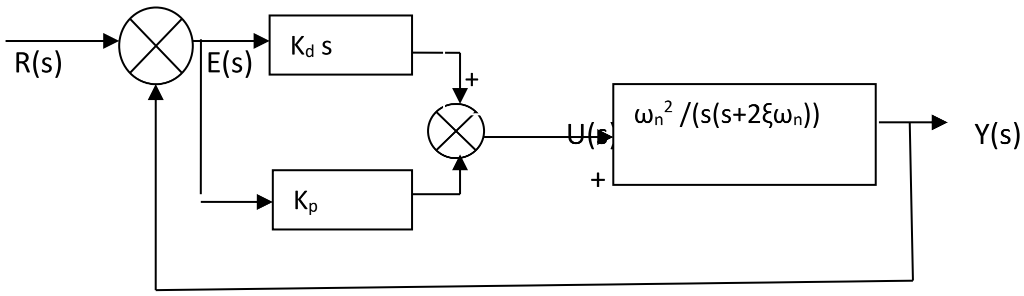


Fig 3.16 PD Controller

Forward path transfer function of the compensated system is:

Y(s)/R(s) = G_c (s) G_p (s) = ω_n^2 (K_p+sK_d)/(s(s+2ξω_n)); which shows that PD controller is equivalent to adding zero at s= -Kp/Kd to the forward path transfer function.

Another way of looking at the PD controller is that since de(t)/dt represents slope of error, the PD controller is essentially an anticipatory control. That is, by knowing the slope, the controller can anticipate direction of error & use it to better control the process. Normally in a linear system, if the slope of e(t) or y(t) due to step input is large, a overshoot will subsequently occur. The derivative control measures the instantaneous slope of e(t), predicts the large overshoot ahead of time, and makes a proper corrective effort before the excessive overshoot actually occurs. The phase lead property may be used to improve the phase margin of the control system.

Advantages of PD Controller:

(a) Improves damping and reduces maximum overshoot.
(b) Reduces rise time and settling time.
(c) Increases bandwidth.

(d) Improves gain margin (GM), phase margin (PM) & Mr.

Problem with derivative control:

(a) May pass noise at higher frequencies.
(b) Not effective for lightly damped or initially unstable system.
(c) May require a large capacitor in circuit implementation.

3.4 Lead, lag, lead-lag, wash-out, notch filters/networks-properties-effect on transfer function, stability, robustness-relative merits:

3.4.1 Lead Compensator: Lead compensators are generally used to **quicken the system response** by increasing natural frequency and/or decreasing the time constant. Lead compensators also **increase the overall stability** of the system. A lead compensator has the general form

TF $_{\text{lead compensator}} = \frac{b(s+a)}{a(s+b)}$; a < b

The b/a simply keeps the steady-state value of the compensator as one. The practical limit in choosing the poles and zeros for the lead compensator is b < 10a. A common application of lead compensator is to cancel a pole at s = -a, which is slowing the time response or causing the system to be unstable. A washout filter is a special case of a lead compensator. Implementation of lead compensator using passive RC network is shown in Fig 3.17.

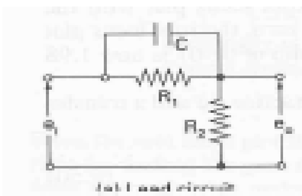


Fig 3.17: Lead Circuit

The movement of the compensator pole and zero is achieved by proper selection of the components in the electrical circuit (R1, R2 and C in fig 3.17).

3.4.2 Lag Circuit: lag compensators are generally used **to slow down the system response** by decreasing natural frequency and/or increasing time constant. They also tend to **decrease the overall stability** of the system. Lag compensation may **also reduce the steady-state error** of a system. A lag compensator has the general form:

TF $_{\text{lag compensator}} = \frac{b(s+a)}{a(s+b)}$; a> b

With lag compensation, a pole is added to the right of a zero. The pole may be used to cancel a zero. A lag circuit using passive RC network is shown in Fig 3.18 below.

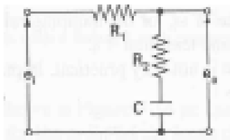


Fig 3.18: A lag compensator

3.4.3 Lead-lag compensator: the combined benefit of lead compensator and lag compensator may be realized using lead-lag compensation. A lead-lag compensator has the general form

$$TF_{\text{lead-lag compensator}} = \frac{bd(s+a)(s+c)}{ac(s+b)(s+d)} \quad ; a > b;\ a < c;\ c < d$$

The (s+a)/(s+b) component represents the lag filter, and the (s+c)/(s+d) component represents the lead filter. Common use of lead-lag compensator is the **attenuation of a specific frequency range** (sometimes called a notch filter). For example, an aircraft structural resonant frequency can be filtered out with a lead-lag compensator if a feedback sensor is erroneously affected by that frequency. For the case where both the transients and steady response are **unsatisfactory** a lead-lag compensator can be used. Fig 3.19 shows electrical circuit that could be used to create lag-lead compensator.

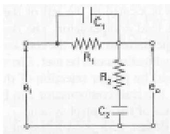


Fig 3.19: Lag-lead Compensator

3.4.4 Washout Filter: another type of high-pass filter is used commonly in aircraft stability augmentation system-a washout filter. It is a special case of lead compensator where the zero is actually a differentiator. It has the form; $G_c(s) = K_{w0}\, s/(s+b)$

In washout compensator low-frequency signals are attenuated, or washed out. Only changes in the inputs are passed through. This is valuable for aircraft feedback control because feeding back a parameter such as roll rate with a wash out filter will not affect the steady state roll rate. Without a wash out filter, the SAS system would constantly oppose the roll rate and decrease the aircraft performance. The gain for high frequency is determined by the corner frequency and the washout filter gain K_{w0} . Additionally, the phase lead is added at lower frequencies.

3.4.5 Notch Filter: A notch filter is a special case of lead-lag compensation or High-low-pass filter. It attenuates a very small frequency range. Typically, these filters are used to take out frequencies that may cause excitation of different aircraft dynamic modes. Transfer function is discussed under lead-lag compensator.

3.5. Adaptive control: Adaptive control is the method used by a controller which must adapt to a controlled system with parameters which vary, or are initially uncertain. For example, as an aircraft flies, its mass will slowly decrease as a result of fuel consumption; a control law is needed that adapts itself to such changing conditions. Adaptive control is different from robust control in that it does not need a priori information about the bounds on these uncertain or time-varying parameters; robust control guarantees that if the changes are within given bounds the control law need not be changed, while adaptive control is concerned with control law changing themselves. **Implementation**: The foundation of adaptive control is parameter estimation. Common methods of estimation include recursive least squares method. This method provides update laws which are used to modify estimates in real time (i.e. as the system operates). It is also called adjustable control. Following are adaptive control techniques.

(a) Feed forward Adaptive Control. **(b) Feedback Adaptive Control**.

Also, there are two methods for Adaptive control implementation:

(a) Direct method **(b) Indirect Method**.

Direct methods are ones wherein the estimated parameters are those directly used in the adaptive controller. In contrast, indirect methods are those in which the estimated parameters are used to calculate required controller parameters. There are different categories of feedback adaptive control:

(a) Adaptive pole placement **(b) Gain scheduling**. **(c) Model Reference Adaptive Controllers**. Block diagram of **Model Identification Adaptive Control** is shown in Fig 3.5.1 below:

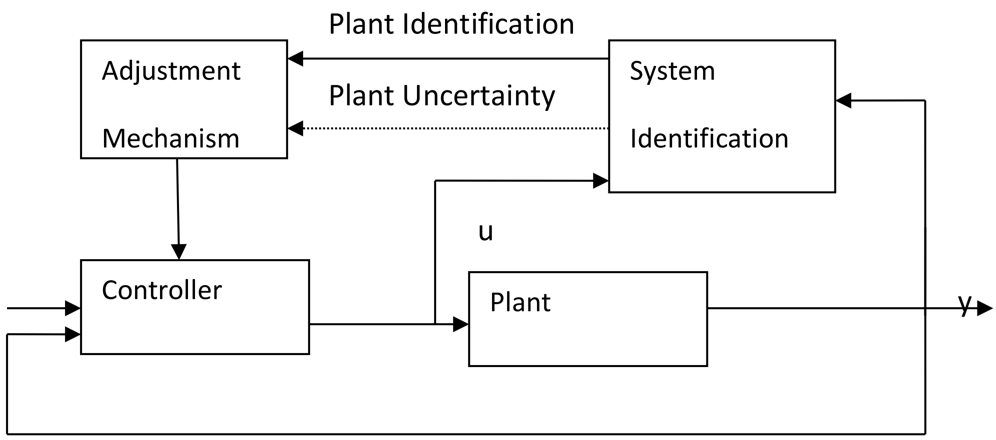


Fig 3.5.1: Model Identification Adaptive Control System

Gain Scheduling: In control theory, gain scheduling is an approach to control non- linear system that uses a family of linear controllers, each of which provides satisfactory control for a different operating point of the system. One or more observable variables, called the scheduling variables, are used to determine what operating region the system is currently in and to enable the appropriate linear controller. For example, in an aircraft flight control system, the altitude and Mach number might be scheduling variables with different linear controller parameters (& automatically plugged into the controller) for various combinations of these two variables.

3.6. Design of a Feedback Controllers: Although series controllers are most common because of their simplicity in implementation, depending on the nature of the system, sometimes there are advantages in placing a controller in a minor feedback loop as shown in fig 3.6.1.

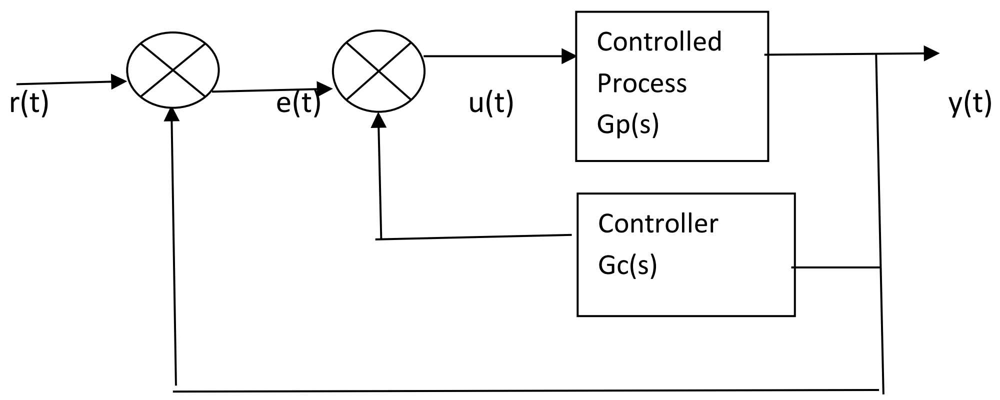


Fig 3.6.1: A feedback controller

For example, a tachometer may be coupled directly to a dc motor not only for the purpose of speed indication, but more often improving the stability of the closed loop system by feeding back the output signal of the tachometer. In principle, the PID controller or phase-lead and phase- lag controllers can all, with varying degree of effectiveness, be applied as minor-loop feedback controllers. Under certain conditions, minor-loop control can yield systems that are more robust, that is, less sensitive to external disturbance or internal parameter variations.

Rate-Feedback or Tachometer-Feedback Control: The principle of using the derivative control of the actuating signal to improve the damping of a closed-loop system can be applied to the output signal to achieve a similar effect. In other words, the derivative of the output signal is fed back and added algebraically to the actuating signal of the system. In practice, if the output variable is mechanical displacement, a tachometer may be used to convert mechanical displacement into an electrical signal that is proportional to the derivative of the displacement. Fig 3.6.2 shows the block diagram of a control system with a secondary path that feeds back the derivative of output.

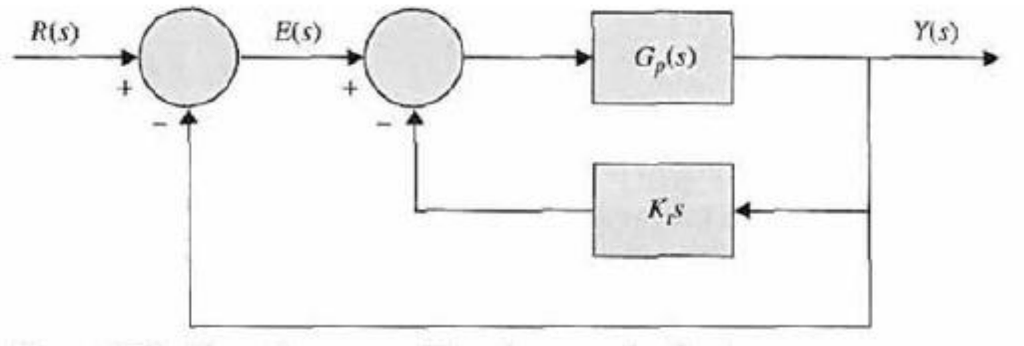


Fig 3.6.2: Control system with tachometer feedback.

Transfer function of tachometer is denoted by K_t s , where Kt is tachometer constant, usually expressed in volts/radian per second for analytical purpose.

Feedback compensation can be used to improve the damping of the system by incorporating an inner rate feedback loop. The stabilizing effect of the inner loop rate feedback can be demonstrated by a simple example. Suppose we have second- order system shown in fig 3.6.3. The amplifier gain can be adjusted to vary the system response. The closed loop transfer function for this system is given by

$M(s) = k_a \omega_n^2 / (s^2 + 2 \xi \omega_n s + k_a \omega_n^2)$

Now we add an inner rate feedback loop as shown in fig 3.6.4, the closed loop transfer function can be obtained as follows. The inner loop transfer functions are

$G_1(s) = \omega_n^2 / (s(s+2 \xi \omega_n))$;$H_1(s) = k_r$ s

Which can be combined as $M(s)_{I.L} = \frac{G1(s)}{1+G1(s)H1(s)} = \omega_n^2 / (s^2 + s\,(2 \xi \omega_n + k_r \omega_n^2))$

The closed loop transfer function can be obtained by letting $G(s)_2 = k_a \omega_n^2 / (s^2 + (2 \xi \omega_n + k_a \omega_n^2)\, s)$. & $H_2(s) = 1$.

This can be combined as:

$M(s)_{O.L.} = \frac{G2(s)}{1+G2(s)H2(s)} = = k_a \omega_n^2 / (s^2 + s\,(2 \xi \omega_n + k_r \omega_n^2) + k_a \omega_n^2)$

If we compare the closed loop-loop transfer function for the cases with and without rate feedback we observe that in the closed loop characteristic equation the damping has been increased. The gain k_r can be used to increase the system damping.

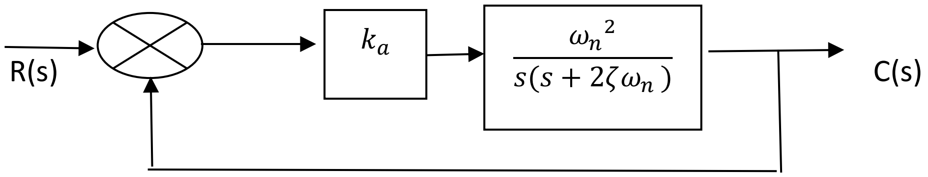


Fig 3.6.3: A second order system

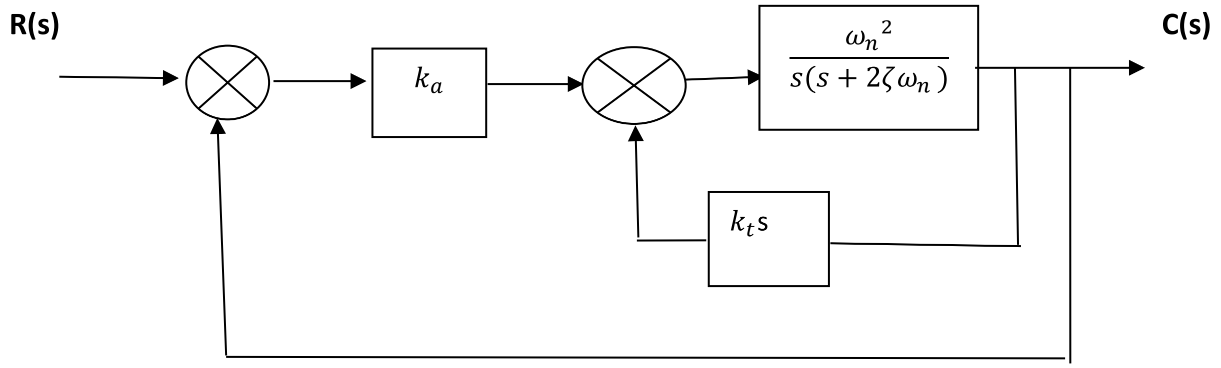


Fig 3.6.4: A second order system with a rate feedback

3.6.2 Significance of Loop Transfer function and Loop Gain: A closed loop control system is shown in fig 3.6.5.

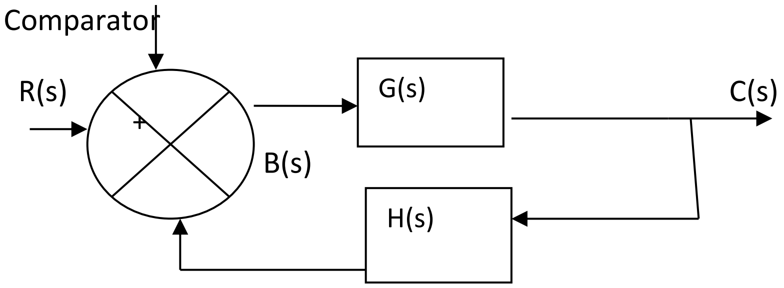


Fig 3.6.5: A closed-loop Control System

R(s) = reference input

C(s) =output signal

B(s) = Feedback signal

E(s) = error signal

G(s) = Open loop transfer function

H(s) = Feedback transfer function

G(s) H(s) = Loop transfer function.

Overall transfer function of the closed loop system is

M(s) =$\frac{G(s)}{1+G(s)H(s)}$

Denominator, 1+ G(s) H(s) = 0, is called characteristic equation. G(s) H(s) is loop transfer function (L(s)). Loop transfer function plays important role in design and performance analysis of control loop system. It determines absolute stability of system, steady state error, and time domain and frequency domain specifications. If we replace s by jω we get loop gain at frequency ω as $|G(j\omega)H(j\omega)|$. Phase angle is denoted by $\lfloor G(j\omega)(H(j\omega)$. When gain becomes unity and phase angle becomes 180° system becomes unstable. Elements in the feedback could be a controller like tachometer or a PID controller.

3.7. Stability of closed Loop System- Frequency response methods and root Locus Methods of analysis, and compensation:

3.7.1 Stability of a closed loop system- Frequency response methods, Gain Margin, Phase Margin-interpretation, significance: The overall transfer function of a control system is given by

M(s) = G(s)/ (1+G(s) H(s))

To find if the closed loop system is stable, we must determine whether F(s) = 1+G(s) H(s) has any root in the right half of the s-plane. For this purpose we can solve the characteristic equation and find its roots. We can also use Routh-Hurwitz criterion to check the number of roots which lie on the right half of the s-plane. In frequency response method we can use **Bode Plot**, **Root locus technique** and/or **Nyquist** criterion to determine the relative stability of the system in terms of **gain margin (GM) and phase margin (PM).**

(a) **Bode plot for determining the stability of a control system**. We know that a Bode plot consists of loop gain in dB vs logarithm of frequency ω and phase angle Vs logarithm of frequency ω. From these two plots we can determine gain cross over and phase cross over points. The gain cross over point on the frequency plot of L(jω) [L(jω) = G(jω) * H(jω)] is a point at which magnitude of L(jω) = 1 or $|G(j\omega)H(j\omega)|dB$ = 0 dB. The frequency at the gain cross over point is called gain cross over frequency. Similarly phase crossover point on the frequency domain plot of L (jω) is a point at which phase angle of L(jω) = 180° . The frequency at the cross over point is called the phase cross over point. From these points we can determine the gain and phase margin.

Gain Margin: The gain margin is defined as the additional gain required for making the system just unstable. It may be expressed either as a factor or in dB.

The phase margin: It is defined as the additional phase lag required for making the system just unstable. It is expressed in degrees.

This is shown in Fig 3.7.1 below:

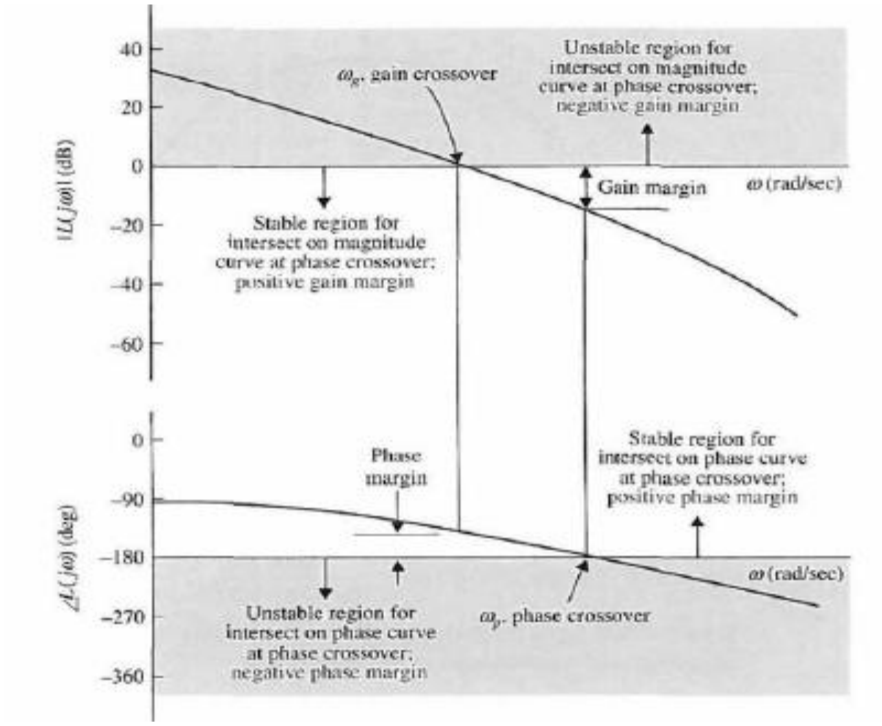


Fig 3.7.1: Gain and Phase Margin from Bode Plot.

(b) **Root Locus Methods of Analysis and Compensation**. In designing a control system, it is desirable to investigate the performance of a control system when one or more parameters are varied. Characteristic equation plays an important role in the dynamic behavior or aircraft motion. The same is true for linear system. In control system design a powerful tool is available for analyzing the performance of a linear system. Basically, the technique provides graphical information in the s-plane on the trajectory of the roots of the characteristic equation for variations in one or more of the system parameters. Typically, most root locus plots consist of only one parameter variation. The Root Locus was introduced by W.R. Evans in 1949. The method allows the control engineer to obtain accurate time-domain response as well as frequency response information of closed loop control system.

Recall the closed loop transfer function of a feedback control system is given as

$$C(s)/R(s) = \frac{G(s)}{1+G(s)H(s)} \qquad (1)$$

The characteristic equation of the closed loop system is found by setting the denominator of the transfer function to zero.

1+G(s) H(s) = 0 (2)

The loop transfer function G(s) H(s) can be expressed in the factored form as follows

G(s) H(s) = $\frac{k(s+z1)(s+z2)\ldots(s+zm)}{(s+p1)(s+p2)\ldots(s+pn)}$

Where z's, p's & k are zeros, poles & gain of the transfer function. The zeros are the roots of the numerator and poles are the roots of the denominator of the loop transfer function. As stated earlier, the root locus is graphical presentation of the trajectory of the roots of the characteristic equation or the poles of the closed loop transfer function for variation of one of the system parameters. Let us examine the root locus plot for the above equation as k is varied. The characteristic equation can be written

G(s) H(s) = -1 (3)

Or $\frac{k(s+z1)(s+z2)\ldots(s+zm)}{(s+p1)(s+p2)\ldots(s+pn)}$ = -1 (4)

For the case k=0, the points on the root locus plots are the poles of the loop transfer function G(s) H(s).

On the other hand for k→ ∞, the points on the root locus are zeros of the loop transfer function. Thus we see that roots of the closed loop transfer function migrate from the poles to the zero of the loop transfer function as k is varied from 0 to ∞ . Furthermore, the points on the root locus for intermediate values of k must satisfy the equation

$\frac{[k][s+z1][s+z2]\ldots[s+zm]}{[s+p1][s+p2]\ldots[s+pn]}$ = 1

And $\sum_{i=1}^{m} \angle s+zi$ - $\sum_{i=1}^{n} \angle s+pi$ = $(2q+1)\pi$; where q = 0, ±1, ±2 … all integers.

Example of Root Locus Plot of a second order System. Fig 3.7.2 below shows block diagram of a second order system.

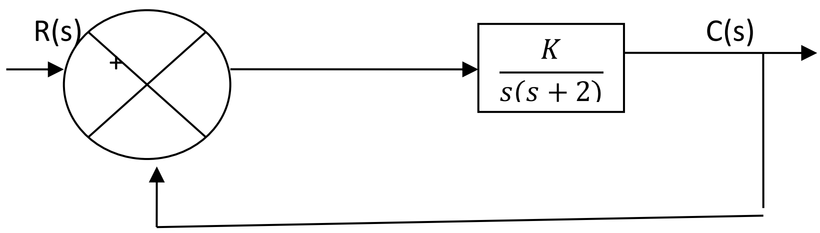


Fig 3.7.2: A second order control system

The root locus diagram gives the roots of the closed loop characteristic equation as k is varied from 0 to ∞ . When k = 0, roots are located at the origin & s= -2. As, k is increased, the roots move along the real axis towards one another until they meet at s= -1. Further increase in k causes the roots to be complex and they move away from the real axis along a line perpendicular to the real axis. When the roots are complex, system is under damped and a measure of the system damping is obtained by measuring the angle drawn from the origin to the point on the complex portion of the root locus. The system damping ratio is given by : $\zeta = \cos(\theta)$

The roots of the characteristic equation can be obtained by root solving algorithm that can be coded on digital computer (Like MATLAB). In addition, there is simple graphical technique that can be used to rapidly construct the root locus diagram of a control system. Root locus of the control system shown in fig 3.7.1 is drawn in fig 3.7.2 below.

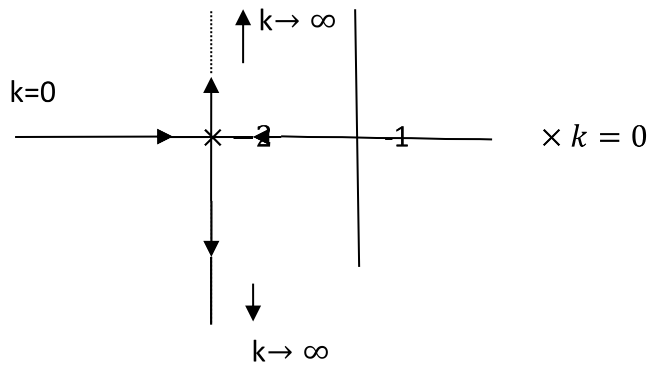


Fig 3.7.2 Root locus diagram of a second order system, whose forward path TF = k/(s(s+2))

3.8 Nyquist's Criterion-stability margin, gain margin, phase margin, interpretation, significance.

3.8.1 Nyquist Criterion: Let's suppose we have a basic feedback system, with transfer function F(s) G(s)/(1+G(s) H(s)). F(s) now is called close loop transfer function. Also, **G(s) is the feed forward transfer function** and **G(s) H(s) is the loop transfer function**. We can make a Nyquist diagram of the loop transfer function G(s) H(s). This is done by replacing s by jω and plotting G (jω) H (jω) on polar plot by varying ω from 0 to∞. The **Nyquist stability criterion** now tells us something about the stability of the entire closed loop transfer function F(s).

First, we need to count the number of poles k of the transfer function G(s) H(s) with real part bigger than zero. (i.e. the number of poles in the right half plane).) Second, we need to count the number of net counterclockwise encirclements of the point -1 of the Nyquist diagram of G(s) H(s). If this number is equal to the number k, then the closed loop system is stable. Otherwise, it is unstable.

3.8.2 Stability margin, gain margin, phase margin, interpretation, significance. In practical situations, in addition to finding out whether a closed loop system is table, if it is also desirable to determine how close it is to instability. This information can be readily determined from the open loop frequency response G (jω) H (jω). The proximity of the open loop frequency response to the point -1+j0 in the GH plane provides a quantitative measure of the relative stability of a closed loop system. Two commonly used measures of relative stability are gain and phase margin. These are defined below:

(a) Gain Margin (GM). The gain margin is defined as the additional gain required for making the system just unstable. It may be expressed either as a factor or in decibels. GM is one of the most frequently used criterions for measuring the relative stability of control system. In the frequency response analysis, gain margin is used to indicate the closeness of the intersection of the negative real axis made by the Nyquist plot of loop transfer function G(jω) H(jω) to the (-1, j0) point. Before defining gain margin, let us first define the phase crossover on the Nyquist plot and the phase-crossover frequency.

Phase Crossover. A phase crossover on the loop transfer function plot is a point at which the plot intersects the negative real axis.

Phase-Crossover Frequency: The phase-crossover frequency ω_p is the frequency at the phase cross over, or we write

$\lfloor L(j\omega)$ = 180°

Gain margin of the closed loop system that has L(s) as its loop transfer function is defined as

Gain margin = GM = 20 $\log_{10} \frac{1}{|L(j\omega p)|}$ = -20 $\log_{10}|L(j\omega)|$ dB

Gain margin is illustrated in the fig 3.8.1 below

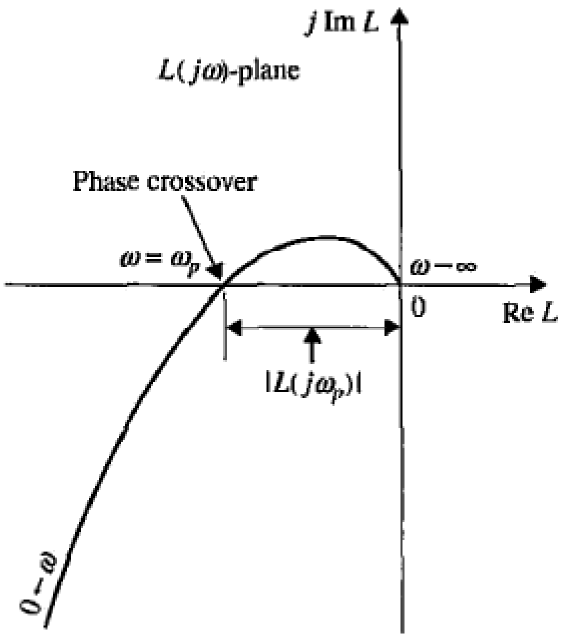


Fig 3.8.1: Definition of the gain margin in the polar coordinates.

(b) Phase Margin. The phase margin is defined as the additional phase lag required for making the system just unstable. Gain margin alone is inadequate to indicate relative stability when system parameters other than loop gain are subject to variation. For example the two systems represented by L(jω) plots in fig 3.8.2 apparently have the same gain margin. However, locus A actually corresponds to a more stable system than locus B, since any change in system parameters that affect the phase of L (jω), locus B may easily be altered to enclose (-1,j0) point. Furthermore, e can show that system B actually has a larger Mr, than system A. Let us first define gain crossover and gain-crossover frequency.

Gain Crossover. The gain crossover is a point on the L(jω) plot at which the magnitude of L(jω) is equal to 1.

Gain-crossover frequency: The gain cross-over frequency, ω_g is the frequency of L (jω) at the gain crossover, or where

$|L(j\omega g|$ = 1

The definition of phase margin is stated as:

Phase margin is defined as the angle in degrees through which the L(jω) plot must be rotated about the origin so that the gain crossover passes through the (-1,j0) point.

Phase margin (PM) = $\lfloor L(j\omega g)$ - 180°

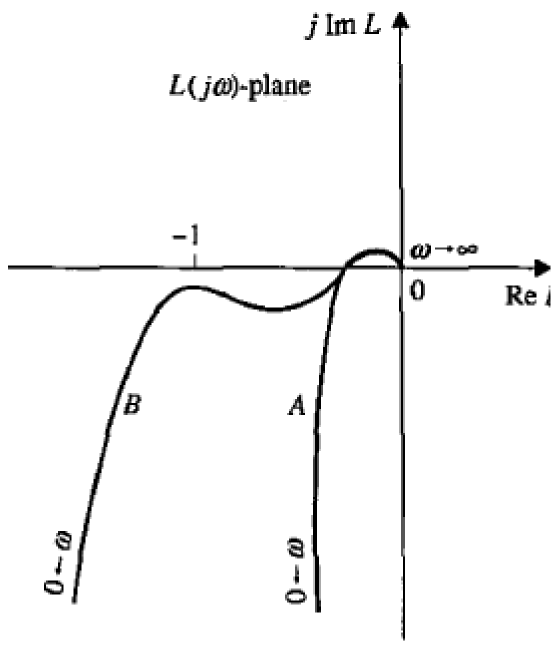


Fig 3.8.2: Two systems having same gain margin but different relative stability.

Fig 3.8.3 shows the Nyquist plot of a typical L(jω) plot, and the phase margin is shown as the angle between the line that passes through the gain crossover and the origin and the negative real axis of the L(jω)-plane. Phase margin is the amount of pure phase delay that can be added to the loop before the closed-loop system becomes unstable.

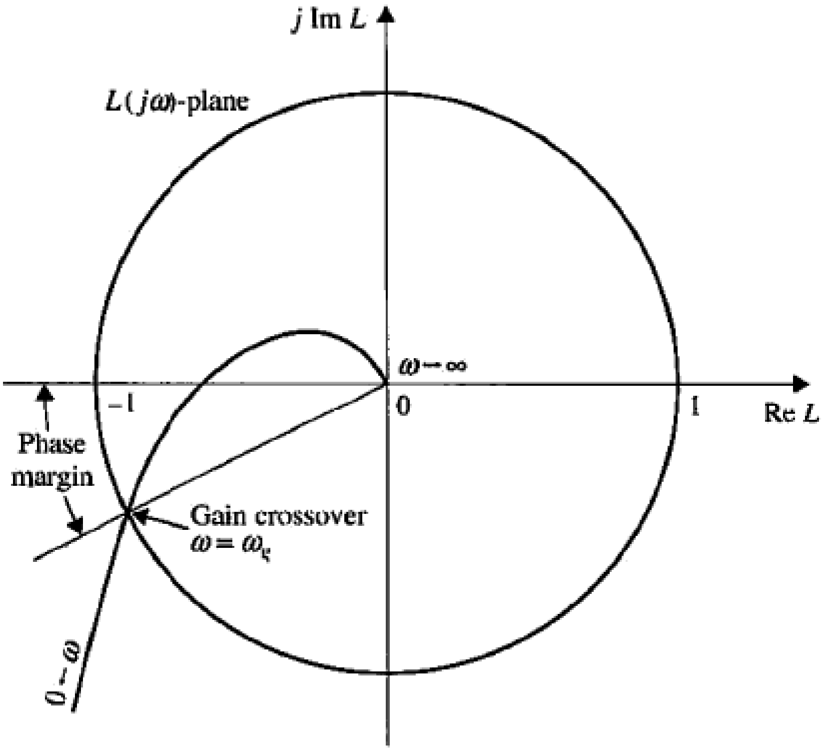


Fig 3.8.3: Phase margin defined in the L(jω) –plane

3.10 Design of a multi loop feedback systems using Root Locus Technique: Design of a multi loop feedback system is explained with an example of a pitch attitude hold auto pilot of a transport aircraft. Basic block diagram is shown in fig 3.10.1

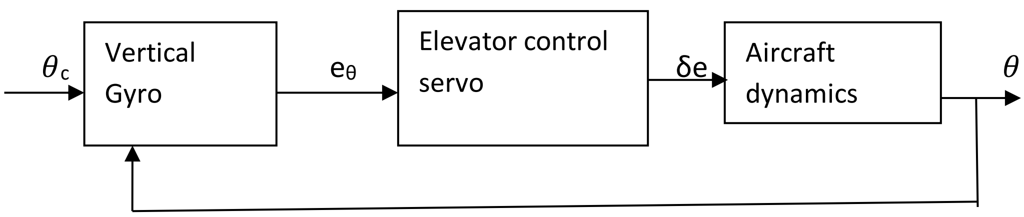


Fig 3.10.1: Block diagram of a pitch displacement auto pilot.

To design the control system for this auto pilot we need the transfer function of each component. The transfer function of the elevator servo can be represented as a first order system

θe/v =1/(τs+1)

where δe,v and τ are the elevator deflection angle, input voltage, and servo motor time constant. The time constant can be assumed to be 0.1s. we can represent the aircraft dynamics by short-period approximation. The short period TF for the business jet aircraft can be shown to be

$$\frac{\Delta\theta}{\Delta\delta e} = -2.0(s + 0.3)/(s(s^2+0.65s+2.15))$$

The fig 3.10.2 is the block diagram representation of the auto pilot.The problem now is one of determing amplifier gain ka so that the control system will have the desired performance. Selection of ka can be determined using a root locus plot of transfer function. Fig 3.10.3 is the root locus plot for the business pitch control auto pilot. As the gain is increased from zero, the system damping decreases rapidly and the system becomes unstable. Even for low values k_a, the system damping would be too low for satisfactory dynamic performance. The reason for poor performance is that the airplane has very little natural damping. To improve the design we could increase the damping of the short period mode by adding an inner loop feedback loop. Fig 3.10.6 is a block diagram representing of displacement auto pilot with pitch rate feed back for improved damping. In the inner loop the pitch rate is measured by a rate gyro and fed back to be added with error signal generated by the difference in pitch attitude. For this problem we have two parameters to select, namely the gains ka and krg. The root locus method can be used to pick both parameters. The procedure is essentially a trial-and-error method. First,the root locus diagram is determined for the inner loop; a gyro gain is selected, and then the outer root locus plot is constructed. Root locus diagram for inner loop is shown in Fig 3.10.4. Value of gyro gain k_{rg} was selected as .901 with damping ratio of .808 and overshhot of 2%. With this value of krg, root locus for outer loop was drawn as shown in fig 3.10.5. Gain k_a was choosen which gives the damping of .776.

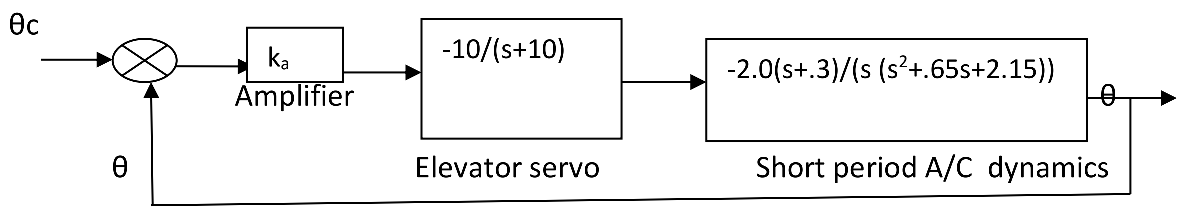


Fig 3.10.2 Block diagram of a pitch displacement auto pilot for a business jet

Root locus in fig 3.10.3 was obtained using MATLAB. Code is shown below:

```
% Root locus of busines jet

s= tf('s');   Gservo= -10/(s+10);   % servo transfer function

Gaircraft= -2.0*(s+0.3)/(s*(s^2+.65*s+2.15));      % aircraft transfer function
```

Gs=Gservo * Gaircraft;   % servo and aircraft in cascade

rlocus(Gs);   grid on

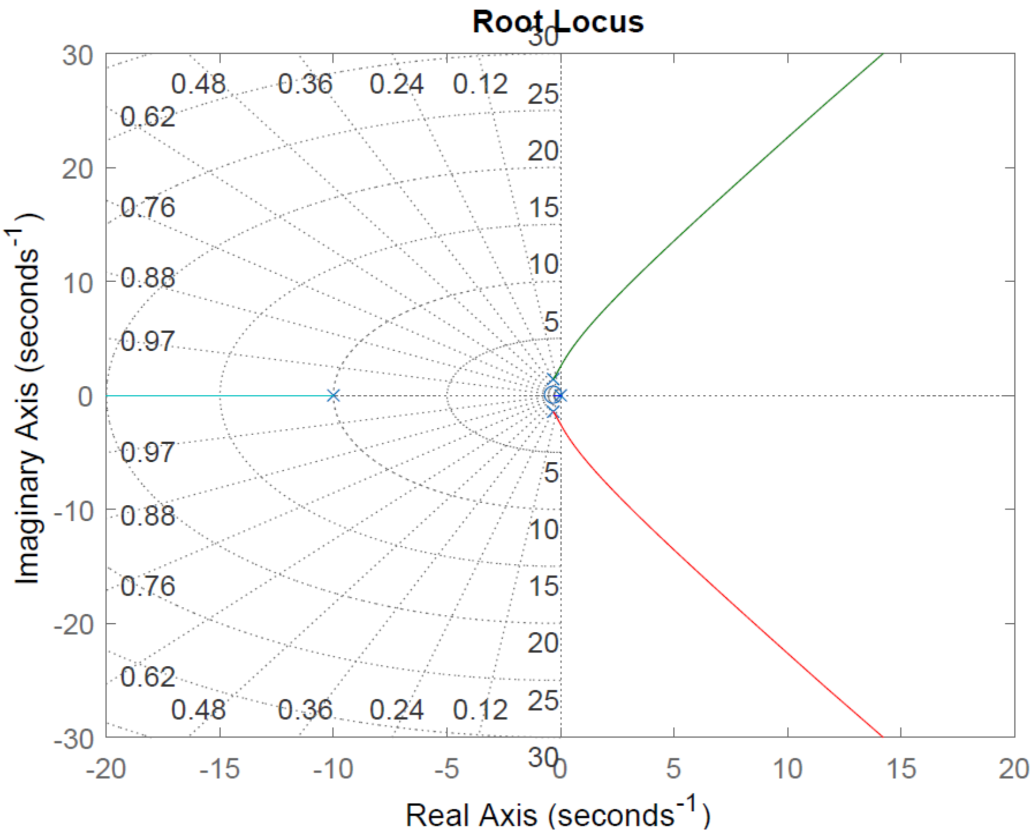


Fig. 10.3: Root Locus diagram of Business Jet

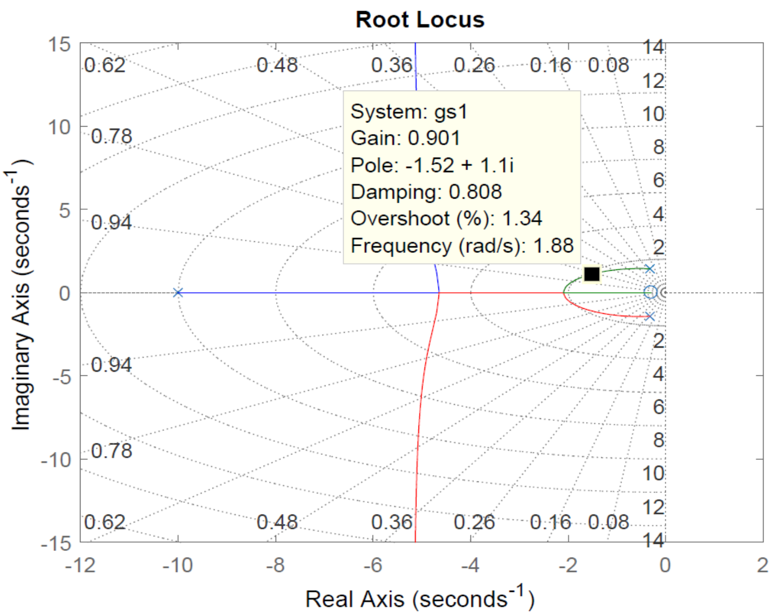


Fig 3.10.4 root Locus of inner loop using rate gyro.

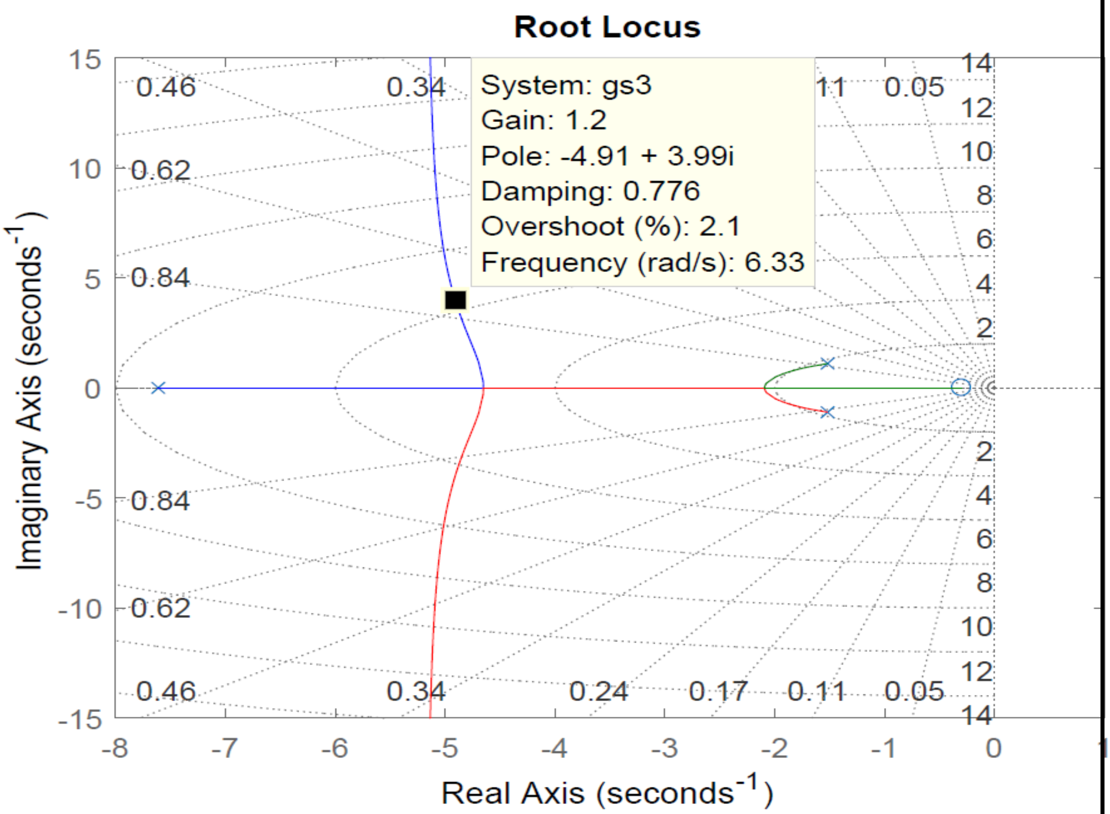


Fig 3.10.5. Root locus of outer loop.

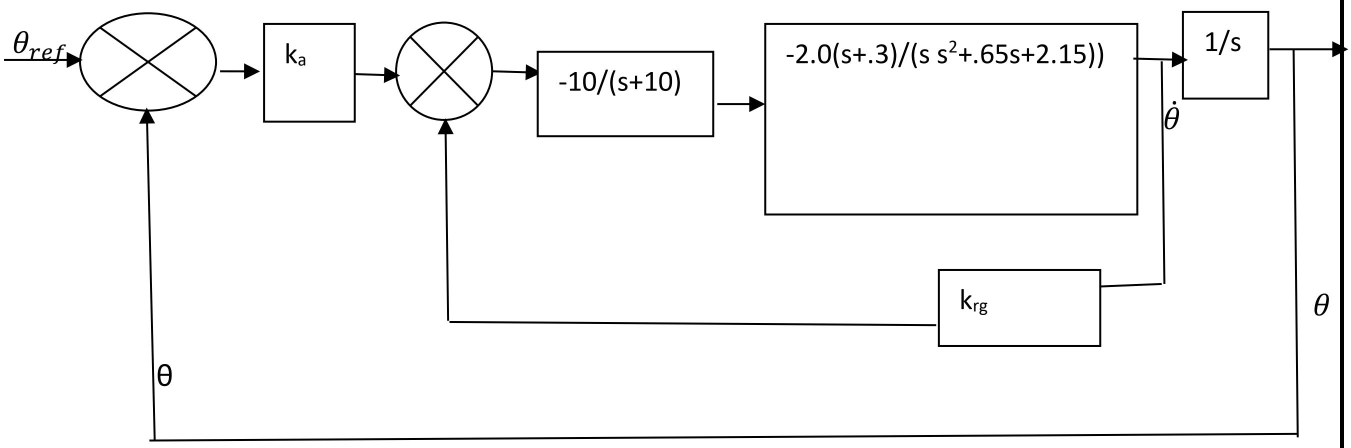


Fig 3.10.6 Block diagram with pitch rate feedback

UNIT-IV

Aircraft Response to Controls-Flying Qualities-Stability and Control Augmentation-Autopilots

Index

4.1 Approximation to aircraft transfer functions. The longitudinal and lateral equations of motion are described by a set of linear differential equations. The transfer function gives the relationship between the output and input to a system. The transfer function is defined as the Laplace transform of output to the Laplace transform of input, with all initial conditions set to zero. Following assumptions are made in **approximation to aircraft transfer functions**.

(a) We assume that aircraft motion consists of small deviations from its equilibrium flight conditions.

(b) We assume that the motion of the aircraft can be analyzed by separating the equation into Longitudinal and Lateral motion (later consists of yawing motion and roll motion).

4.1.1 Longitudinal Transfer Function Approximations: The longitudinal motion of an airplane (controls fixed) disturbed from its equilibrium flight condition is characterized by two oscillatory modes of motion. Fig 4.1 below illustrates these basic modes. We see that one mode is lightly damped & has a long period. This motion is called the long-period or phugoid mode. It occurs at constant angle of attack. The second basic mode is heavily damped & has a very short period & it is appropriately called the short-period mode.

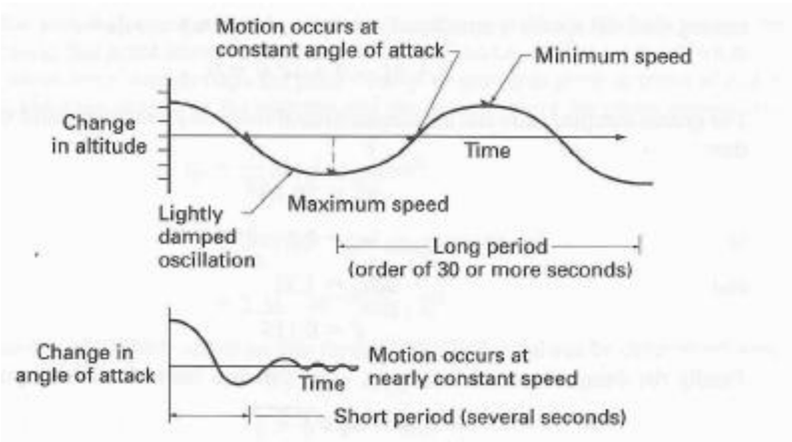


Fig 4.1 Phugoid and Short Period Oscillations

Longitudinal differential equations can be written as:

$$\left(\frac{d}{dt} - X_u\right)\Delta u - X_w + (\text{g Cos } \theta_0)\Delta\theta = X_\delta\Delta\delta + X_{\delta_T}\,\Delta\delta_T$$

$$-Z_u\Delta u + \left[(1 - Z_{\dot{w}})\frac{d}{dt} - Z_w\right]\Delta w - \left[(u_0 + Z_q)\frac{d}{dt} - g\ sin\theta_0\right]\Delta\theta = Z_\delta\Delta\,\delta + Z_{\delta_T}\,\Delta\delta_T$$

$$-M_u\Delta u - \left(M_{\dot{w}}\frac{d}{dt} + M_w\right)\Delta w + \left(\frac{d^2}{dt^2} - M_q\frac{d}{dt}\right)\Delta\theta = M_\delta\Delta\delta + M_{\delta_T}\,\Delta\delta_T$$

Where $\Delta\delta$ and $\Delta\delta_T$ are the aerodynamic and propulsive controls, respectively. If we take the Laplace transform of above equations and divide by control deflection we can find the transfer function $\Delta u/\Delta\delta$, $\Delta\theta/\Delta\delta$, $\Delta//\Delta\delta$. These equations can be solved by Cramer's Rule to find the transfer functions.

Transfer functions can be expressed as two polynomials

$$\Delta u/\Delta\delta = \frac{A_u\, s^3 + B_u\, s^2 + C_u\, \mathrm{s} + \mathrm{Du}}{\Delta_{longitude}}$$

$$\Delta w/\Delta\delta = \frac{A_W\, s^3 + B_W\, s^2 + \mathrm{Cw\ s} + \mathrm{Dw}}{\Delta_{longitude}}$$

$$\Delta\theta\ /\Delta\delta = \frac{A_\theta\, s^2 + B_\theta\, \mathrm{s} + \mathrm{C}_\theta}{\Delta_{longitude}}$$

$\Delta_{longitude}$ = $As^4 + Bs^3 + Cs^2 + Ds + E$

State Variable Representation of Equation of Motion: When equations are written as a system of first-order differential equations, they are called state space or state variable equations and expressed mathematically as

$\dot{x}$= Ax + Bη; where x is the state vector and η is control vector & the A & B contain the aircraft's dimensional stability derivative. The above differential equations of longitudinal motion can be further simplified as follows.

In practice, the force derivatives Z_q and $Z_{\dot{w}}$ usually are neglected because they contribute very little to the aircraft response. Therefore too simplify our presentation of the equations of motion in the state-space form we will neglect both these derivatives. Rewriting the equations in the state-space form

$$\begin{bmatrix} \Delta_{\dot{u}} \\ \Delta_{\dot{w}} \\ \Delta_{\dot{q}} \\ \Delta_{\dot{\theta}} \end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u + Z_u M_{\dot{w}} & M_w + Z_w M_{\dot{w}} & M_q + M_{\dot{w}} u_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} X_\delta & X_{\delta_T} \\ Z_\delta & Z_{\delta_T} \\ M_\delta + Z_\delta M_{\dot{w}} & X_{\delta_T} + M_{\dot{w}} Z_{\delta_T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\delta_T \end{bmatrix}$$

Where the state vector x and control vector η are given by

x = $\begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$, η = $\begin{bmatrix} \Delta\delta \\ \Delta\delta_T \end{bmatrix}$; and the matrices A and B are given by

$$A = = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u + Z_u M_{\dot{w}} & M_w + Z_w M_{\dot{w}} & M_q + M_{\dot{w}} u_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B=\begin{bmatrix} X_\delta & X_{\delta_T} \\ Z_\delta & Z_{\delta_T} \\ M_\delta + Z_\delta M_{\dot{w}} & X_{\delta_T} + M_{\dot{w}} Z_{\delta_T} \\ 0 & 0 \end{bmatrix}$$

4.1.1. Phugoid Mode Approximation: In this mode there is no change in angle of attack.

$\Delta\alpha = \frac{\Delta w}{u_0}$ $\Delta\alpha$ = 0→ $\Delta w = 0$; Making these assumptions, the homogeneous longitudinal state equations reduce to the following:

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & -g \\ \frac{-Z_u}{u_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix}$$

The eigenvalues of the long period approximation are obtained by solving the equation

$|\lambda I - A|$ = 0

$$\begin{vmatrix} \lambda - X_u & g \\ \frac{Z_u}{u_0} & \lambda \end{vmatrix} = 0$$

Expanding the determinant yields

$\lambda^2 - X_u\,\lambda - \frac{Z_u g}{u_0} = 0$; or

$$\lambda_p = \left[X_u \pm \sqrt{X_u^2 + 4\frac{Z_u g}{u_0}}\right]/2.0$$

The frequency and damping ratio can be expressed as

$$\omega_{np} = \sqrt{\frac{-Z_u g}{u_0}}$$

$\xi_p = \frac{-X_u}{2\omega_{n_p}}$; If we neglect compressibility effects, the frequency and damping ratios for the long-period motion can be approximated by the following equation:

$$\omega_{n_p} = \sqrt{2}\,\frac{g}{u_0}$$

$$\xi_p = \frac{1}{\sqrt{2}}\,\frac{1}{L/D}$$

Notice that the frequency of oscillation and the damping ratio are inversely proportional to the forward speed and the lift-to-drag ratio, respectively. We see from this approximation that the phugoid damping is degraded as the aerodynamic efficiency (L/D) is increased. When pilots are flying an airplane under visual flight rules the phugoid damping and frequency can vary over a wide range and they will still find the airplane acceptable to fly. On the other hand, if they are flying the airplane under instrument flight rules low phugoid damping will become very objectionable. To improve the damping of the phugoid motion, the designer would have to reduce the lift-to-drag ratio of the airplane. Because this would degrade the performance of the airplane, the designer would find such choice unacceptable and would look for another alternative, such as an automatic stabilization system to provide the proper damping characteristics.

4.1.2 Short-Period Approximation: An approximation to the short period mode of motion can be obtained by assuming Δu = 0 and dropping the X-force equation. The longitudinal state-space equations reduce to the following:

$$\begin{bmatrix}\Delta\dot{w}\\ \Delta\dot{q}\end{bmatrix}=\begin{bmatrix}Z_w & u_0\\ M_w+M_{\dot{w}}Z_w & M_q+M_{\dot{w}}u_0\end{bmatrix}\begin{bmatrix}\Delta w\\ \Delta q\end{bmatrix}$$

This equation can be written in terms of the angle of attack by using the relationship

$$\Delta\alpha=\frac{\Delta w}{u_0}$$

In addition, one can replace the derivative due to w and $\dot{w}$ with derivative due to α and $\dot{\alpha}$ by using the following equations. The definition of the derivative $M_\propto$ is

$$M\alpha=\frac{1}{I_y}\frac{\partial M}{\partial\propto}=\frac{1}{I_y}\frac{\partial M}{\partial\left(\frac{\Delta w}{u_0}\right)}=\frac{u_0}{I_y}\frac{\partial M}{\partial w}=u_0M_w$$

In a similar way we can show that

$Z_\propto = u_0Z_w$ and $M_{\dot{\propto}} = u_0M_{\dot{w}}$

Using these expressions, the state equations for the short-period approximation can be written as

$$\begin{bmatrix}\Delta\dot{\propto}\\ \Delta\dot{q}\end{bmatrix}=\begin{bmatrix}Z_{\propto/u_0} & 1\\ M_\propto+M_{\dot{\propto}}\frac{Z_\propto}{u_0} & M_q+M_{\dot{\propto}}\end{bmatrix}\begin{bmatrix}\Delta\propto\\ \Delta q\end{bmatrix}$$

The eigenvalues of the state equation can again be determined by solving the equation

$$|\lambda I-A|=0$$

Which yields

$$\begin{vmatrix}\lambda-\frac{Z_\propto}{u_0} & -1\\ -M_\propto-M_{\dot{\propto}}\frac{Z_\propto}{u_0} & \lambda-\left(M_qM_{\dot{\propto}}\right)\end{vmatrix}=0$$

The characteristic equation for this determinant is

$$\lambda^2 - \left(M_q + M_{\dot{\propto}} + \frac{Z_{\propto}}{u_0}\right)\lambda + M_q \frac{Z_{\propto}}{u_0} - M_{\propto} = 0$$

The approximate short-period roots can be obtained easily from the characteristic equation,

$$\lambda_{sp} = \left(M_q + M_{\dot{\propto}} + \frac{Z_{\propto}}{u_0}\right)/2 \ \pm \left[\left(M_q + M_{\dot{\propto}} + \frac{Z_{\propto}}{u_0}\right)^2 - 4\left(M_q \frac{Z_{\propto}}{u_0} - M_{\propto}\right)\right]^{1/2} /2$$

Or in terms of the damping and frequency

$$\omega_{n_{sp}} = \left[\left(M_q \frac{Z_{\propto}}{u_0} - M_{\propto}\right)\right]^{1/2}$$

$$\xi_{sp} = -\left[M_q + M_{\dot{\alpha}} + \frac{Z_{\alpha}}{u_0}\right] /(2(\omega_{n_{sp}})$$

4.1.3. Lateral Approximation of aircraft transfer function. The characteristic equation of aircraft lateral motion is characterized by the following equation.

$A\lambda^4 + B\,\lambda^3 + C\lambda^2 + D\,\lambda + E = 0$

Where A, B, C, D & E are the functions of stability derivative, mass and inertia characteristic of the airplane.

In general we find that the roots of the characteristic equation to be composed of two real roots and fair of complex roots. The roots will be such that the airplane response can be characterized by the following motions.

(a) A slowly convergent or divergent motion, called the spiral mode.
(b) A highly convergent motion, called the rolling mode.
(c) A lightly dumped oscillating motion having a low frequency, called the Dutch roll.

Spiral mode is shown in fig 4.2. Roll mode in fig 4.3 and Dutch roll motion in Fig 4.4.

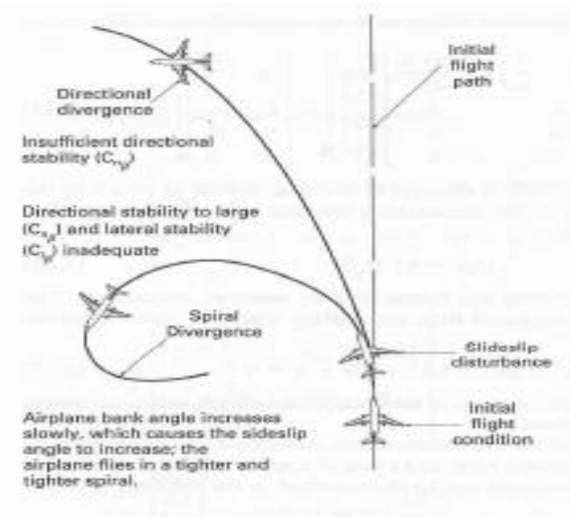


Fig 4.2: Spiral Mode

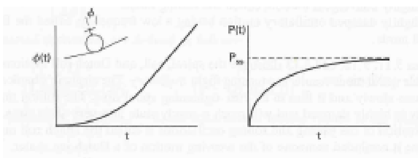


Fig 4.3 Roll Mode.

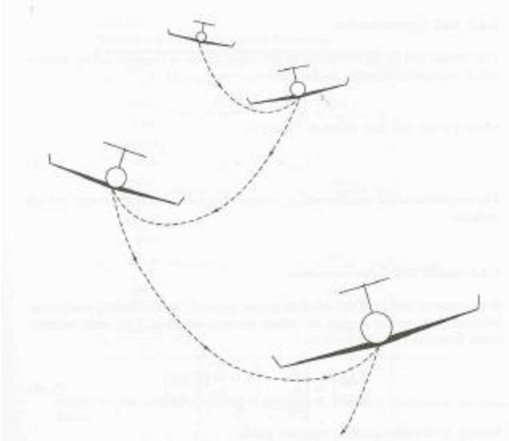


Fig 4.4: Dutch Roll Motion.

(a) **Spiral Approximation**. The characteristic root of the spiral mode is

$\lambda_{spiral} = \frac{L_\beta N_r - L_r N_\beta}{L_\beta}$

The stability derivative L_β (dihedral effect) & N_r (yaw rate damping), are usually negative quantities. On the other hand, N_β (directional stability) & L_r (Roll moment due to yaw rate) are generally positive quantities. Hence condition for stable spiral mode is

$$L_\beta N_r > L_r N_\beta$$

Increasing the dihedral effect L_β and/or the yaw damping can be used to make the spiral mode stable.

(b) **Roll Approximation**: $\lambda_{roll} = L_p = -1/\tau$

The magnitude of roll damping Lp can be determined by the wing & tail surfaces.

(c) **Dutch Roll approximations**: If we consider that Dutch roll consists of side slipping & yawing motions, we get

$$\begin{bmatrix} \Delta\dot{\beta} \\ \Delta\dot{r} \end{bmatrix} = \begin{bmatrix} \frac{Y_\beta}{u_0} & -\left(1 - \frac{Y_r}{u_0}\right) \\ N_\beta & N_r \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta r \end{bmatrix}$$

Solving for the characteristic equation yields

$$\lambda^2 - \left(\frac{Y_\beta + u_0 N_r}{u_0}\right)\lambda + \frac{Y_\beta N_r - N_\beta Y_r + u_0 N_\beta}{u_0}$$

From this expression we can determine the undamped natural frequency and the damping ratio as follows:

$$\omega_{nDR} = \sqrt{\frac{y_\beta N_r - N_\beta y_r + u_0 N_\beta}{u_0}}$$

$\xi_{DR} = \frac{-1}{2\omega_{nDR}}\left(\frac{y_\beta + u_0 N_r}{u_0}\right)$

4.2. Response of aircraft to pilot's control inputs.

4.2.1 Response of aircraft to Pilot's control input: Response of an aircraft to control input or atmosphere can be done by considering step input and sinusoidal input. The step and sinusoidal input functions are important for two reasons. First, the input to many physical systems takes the form of either a step change or sinusoidal signal. Second, an arbitrary function can be represented by a series of step changes or a periodic function can be decomposed by means of Fourier analysis into a series of sinusoidal waves. If we know the response of a linear system to either a step or sinusoidal input, then we can construct the system's response to an arbitrary input by the principle of superposition.

Of particular importance to the study of aircraft response to control or atmospheric inputs is the steady-state response to a sinusoidal input. If the input to a control system is sinusoidal, then after the transients have died out the response of the system also will be sinusoid of the same frequency. The response of the system is completely described by the ratio of the output to input amplitude and the phase difference over the frequency range from zero to infinity. The magnitude and phase relationship between the input and output signals is called the frequency response. The frequency response can be obtained readily from the system transfer function by replacing the Laplace variable s by jω. The frequency response information is usually presented in graphical form using either rectangular, polar, log-log or semi-log plots as discussed in unit-II. Consider the transfer function, given by

$$G(s) = \frac{k(1+T_a s)(1+Tbs)\ldots}{s^m(1+T_1 s)(1+T_2 s)\ldots(1+\frac{2\zeta}{\omega_n}s+\frac{s^2}{\omega_n^2})}$$

Replacing the s by jω and rewriting the transfer function in polar form yields

$$M(\omega) = |G(j\omega)| = \frac{|k| \times |1+T_a j\omega| \times |1+T_a j\omega| \times |1+T_b j\omega| \ldots}{|(j\omega)^m| \times |1+T_1 j\omega| \ldots \left|1-\left(\frac{\omega}{\omega_n}\right)^2+2\zeta\frac{\omega}{\omega_n}j\right| \ldots} \times \exp[j(\varphi_a+\varphi_b \ldots -\varphi_1-\varphi_2\ldots)]$$

Now, if we take the logarithm of this equation, we obtain

$\text{Log } M(\omega) = \log |G(j\omega)| = \log k + \log|1+T_a j\omega| + \log|1+T_b j\omega| \ldots - m\log|j\omega| - \log|1+T_1 j\omega| -$

$$\log|1+T_2 j\omega| - \log\left|1-\left(\frac{\omega}{\omega_n}\right)^2+2\zeta\frac{\omega}{\omega_n}j\right| - \ldots \quad \textbf{(1)}$$

And phase of G (jω); $\lfloor G(j\omega) = \tan^{-1}\omega T_a + \tan^{-1}\omega T_b + \ldots - m\,(90°) - \tan^{-1}\omega T_1 - \ldots \tan^{-1}\left(\frac{2\zeta\omega_n}{\omega_n^2-\omega^2}\right)$ **(2)**

In practice, the log magnitude is often expressed in decibels (dB). The magnitude in decibels is found by multiplying each term in equation (1) by 20:

Magnitude in dB = 20 $\log_{10} |G(j\omega)|$

The frequency response information of a transfer function is represented by two graphs, one of the magnitude and other of the phase angle, both versus the frequency on a logarithm scale. The plots are referred as Bode diagrams after H.W. Bode who made significant contribution to frequency response analysis.

Let us see, how these plots can be used to analyze the response of aircraft to control inputs. Let us consider the longitudinal pitch angle to elevator transfer function that can be shown as indicated below, where the coefficient A_θ and B_θ, and so forth are functions of the aircraft stability derivatives. The longitudinal pitch angle to elevator transfer function is as follows:

$$\frac{\theta(s)}{\delta_e(s)} = \frac{A_\theta s^2 + B_\theta s + C_\theta}{As^4 + Bs^3 + Cs^2 + Ds + E}$$

This can be written in the factored form:

$$\frac{\theta(s)}{\delta_e(s)} = \frac{k_{\theta\delta}\,(T_{\theta 1}s+1)(T_{\theta 2}\,s+1)}{\left(\frac{s^2}{\omega_{nsp}^2} + \frac{2\xi_{sp}}{\omega_{nsp}}s + 1\right)\left(\frac{s^2}{\omega_{np}^2} + \frac{2\xi_{sp}}{\omega_{np}}s + 1\right)}$$

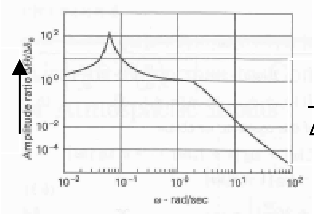
The magnitude and phase angle for the control transfer function is obtained by replacing s by $j\omega$ as follows:

$$\left|\frac{\theta(j\omega)}{\delta_e(j\omega)}\right| = \frac{|k_{\theta\delta}||T_{\theta 1}j\omega+1|}{\left|\frac{(j\omega)^2}{\omega_{nsp}^2} + \frac{2\xi_{sp}}{\omega_{nsp}}j\omega + 1\right|}\;\frac{|T_{\theta 2}j\omega+1|}{\left|\frac{(j\omega)^2}{\omega_{np}^2} + \frac{2\xi_{sp}}{\omega_{np}}j\omega + 1\right|}$$

Phase angle $\angle\frac{\theta(j\omega)}{\delta_e(j\omega)} = \tan^{-1}\omega T_{\theta 1} + \tan^{-1}\omega T_{\theta 2} - \tan^{-1}\left(\frac{2\omega\xi_{sp}\omega_{nsp}}{\omega_{nsp}^2-\omega^2}\right) - \tan^{-1}\left(\frac{2\omega\xi_p\omega_{np}}{\omega_{np}^2-\omega^2}\right)$

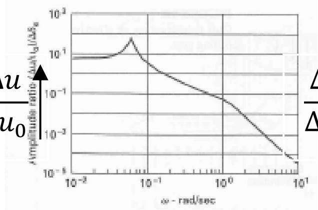
The frequency response for pitch attitude to control deflection for a typical business jet aircraft is shown in fig4.5. The amplitude ratios at both the phugoid and short-period frequencies are of comparable magnitude. At very large frequencies, the amplitude ratio is very small, which indicates that the elevator has negligible effect on the pitch attitude in this frequency range. The frequency response for the change in forward speed and angle of attack is shown in Fig 4.6 and 4.7 respectively. For the speed elevator transfer function the amplitude ratio is large at phugoid frequency and very small at the short- period frequency. It is because short-period motion occurs at essentially constant speed Fig 4.7 shows the amplitude ratio of the angle of attack to elevator deflection; here we see that the angle of attack (AOA) is constant at low frequencies. It is because in Phugoid mode AOA remains constant. The phase plot will show that there is a large phase lag in the response of the speed change to elevator inputs. The phase lags for α/δ is much smaller, which means that the AOA will respond faster than the change in forward speed to an elevator input.

$\frac{\Delta\theta}{\Delta\delta_e}$



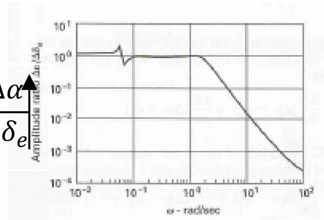
ω - radian/sec

$\frac{\Delta u}{\Delta u_0}$



ω- radian/sec

$\frac{\Delta\alpha}{\Delta\delta_e}$



ω- radian/sec

Fig 4.5 Magnitude plot ($\frac{\Delta\theta}{\Delta\delta_e}$ Vs ω) Fig 4.6 Magnitude plot($\frac{\Delta u}{u_0}$ Vs ω) Fig 4.7 Plot of $\frac{\Delta\alpha}{\Delta\delta_e}$ Vs ω

4.3 The control task of the pilot: The control task of the pilot is to fly the aircraft safely in the assigned mission of the aircraft. For a passenger aircraft mission profile will consist of take-off, cruise and landing at the designated airport. Similarly, a military aircraft being a weapon delivery platform should be able to strike the designated target accurately. To accomplish these missions pilot should be able to control and fly the aircraft accurately and maintain the designated route without fatigue. The aircraft should be controllable even when it is disturbed from its equilibrium position either by pilot's action or by atmospheric turbulence. An airplane must have sufficient stability such that the pilot does not become fatigued by constantly having to control the airplane owing to external disturbance. Although airplanes with little or no inherent aerodynamic stability can be flown, they are unsafe to fly unless they are provided artificial stability by stability augmentation system. Two conditions are necessary for an airplane to fly its mission successfully. The airplane must be able to achieve equilibrium flight and it must have the capability to maneuver for a wide range of flight velocity and altitude. The stability and control characteristic of an airplane are referred to as the vehicle's handling or flying qualities. Airplane with poor handling qualities will be difficult to fly and could be dangerous. An airplane will be considered of poor design if it is difficult to handle regardless of how outstanding the airplane's performance might be.

Precision tasks such as landing approach, tracking, and formation flying in military aircraft can only be accomplished successfully if the aircraft's dynamic stability characteristics are within the acceptable limits. Also pilot's should have sufficient control authority (usually referred to as control power) to trim and maneuver the airplane throughout the flight envelop. Force per g should be uniform throughout the flight envelop.

4.4. Flying qualities of aircraft-relation to airframe transfer function.

4.4.1 Flying Qualities of an Aircraft: The flying qualities of an airplane are related to the stability and control characteristics and can be defined as those stability and control characteristics important in forming the pilot's impression of the aircraft. The pilot forms a subjective opinion about the ease or difficulty of controlling the airplane in steady and maneuvering flight. In addition to the longitudinal dynamics, the pilot's impression of the airplane is influenced by the feel of the airplane, which is provided by the stick force and stick force gradients. The Department of Defense and Federal Aviation Administration has a list of specifications dealing with airplane handling qualities. These requirements are used by the procuring agencies to determine whether an airplane is acceptable for certification. The purpose of these requirements is to ensure that the airplane has flying qualities that place no limitation in the vehicle's flight safety nor restrict the ability of the airplane to perform its intended mission. Military standard MIL-F_875C gives the requirements for military aircraft.

As one might guess, the flying qualities expected by the pilot depend on the type of aircraft and the flight phase. Aircraft are classified according to size and maneuverability. Following are classifications, categories and levels of flying qualities defined as per **MIL-F_875C requirements**.

(a) Classification of airplanes: Airplane can be placed in one of the following classes:

Class I: Small, light airplanes

Class II: Medium weight, low-to-medium maneuverability airplane

Class III: Large, heavy, low-to-medium maneuverability airplanes.

Class IV: High maneuverability airplanes

(b) Flight Phase Category: Flight Phases descriptions of most military airplane mission are:

Category A: Those non- terminal Flight Phases that require rapid maneuvering, precision tracking, or precise flight-path control. Examples are air-to-air combat, ground attack, in-flight refueling, and close formation flying.

Category B: Those non- terminal Flight phases that are normally accomplished using gradual maneuvers and without precision tracking, although accurate flight path control may be required. Examples are climb, cruise, and descent.

Category C: terminal Flight Phases normally accomplished using gradual maneuvers and usually require flight-path control. Examples are take-off, approach, go-around, and landing.

(c) Level of flying qualities: The Levels are:

Level 1: Flying qualities clearly adequate for the mission Flight Phase

Level 2: Flying qualities adequate to accomplish the mission Flight Phase, but some increase in pilot work load or degradation in mission effectiveness, or both, exists.

Level 3: Flying qualities such that the airplane can be controlled safely, but pilot work load is excessive or mission effectiveness is inadequate, or both.

4.4.2 Longitudinal flying qualities- relation to airframe transfer function: Extensive research has been done to relate the flying qualities of airplane with stability and control characteristic of an aircraft. The fig 4.8 shows the relationship between the level of flying qualities and the damping ratio and un-

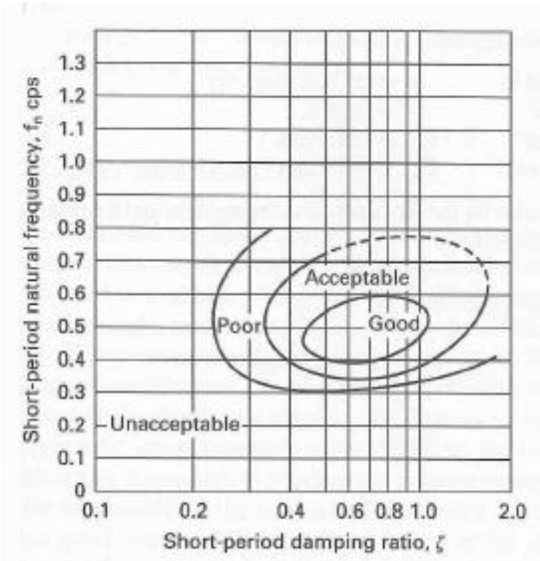


Fig 4.8 Short period flying qualities

damped natural frequency of short period mode.

(a) Phugoid stability. The long period oscillations which occur when the airplane is disturbed from a stabilized airspeed following a disturbance shall meet the following requirements:

Level 1: ξ_{ph} at least 0.04

Level 2: ξ_{ph} at least 0

Level 3: T_2 at least 55 seconds (Where T_2 is time to double amplitude)

(b) Short period damping ratio limits: The equivalent short-period damping ratio, shall be within the limits of table 4.4.2

Table 4.4.2: Short Period Damping Ratio Limits

Level	Category A & C Flight Phase		Category B Flight Phase	
	Minimum	Maximum	Minimum	Maximum
1	0.35	1.30	0.30	2.0
2	0.25	2.00	0.20	2.0
3	0.15	-	0.15	-

4.4.3 Lateral flying qualities- relation to airframe transfer function:

(a) Dutch Roll: The frequency ω_{nd} and damping ratio ζ_d of the lateral-directional oscillations following a yaw disturbance input shall exceed the minimum value given in table 4.4.3

(b) Roll mode: The roll- mode time constant, τ_R, shall be no greater than the appropriate value in table 4.4.4.

Table 4.4.3: Minimum Dutch roll frequency and damping

Level	Flight Phase Category	Class	Minimum ζ_d	Minimum $\omega_{nd}\zeta_d$ rad/s	Minimum ω_{nd} rad/s
1	A [Combat & Ground Attack]	IV	0.4	-	1.0
	A	I, IV II,III	0.19 0.19	0.35 0.35	1.0 0.4
	B	All	0.08	0.15	0.4
	C	I,IV II-Landing, III	0.08 0.08	0.15 0.10	1.0 0.4
2	All	All	0.02	0.05	0.4
3	All	All	0	0	0.4

Table 4.4.4: Maximum roll time constant, seconds

Flight Phase Category	Class	Level 1	2	3
A	I,IV	1.0	1.4	
	II,III	1.4	3.0	
B	All	1.4	3.0	10
C	I,IV	1.0	1.4	
	II-Ldg,III	1.4	3.0	

(c) Spiral Stability: The combined effect of spiral stability, flight-control-system characteristic and rolling moment change with speed shall be such that following a disturbance in bank of up to 20 degrees, the time for the bank angle to double shall be greater than the value in table 4.4.5.

Table 4.4.5: Spiral stability-minimum time to double amplitude

Flight Phase Category	Level 1	Level 2	Level 3
A & C	12 sec	8 sec	4 sec
B	20 sec	8 sec	4 sec

4.5 Pilot's opinion rating: Flying qualities of an airplane is assessed by test pilot's comment obtained from simulations and test flying of the aircraft. A structured rating scale for aircraft handling qualities was developed by NASA in the late 1960s called the Cooper-Harper rating scale. This rating applies to specific pilot-in-loop tasks such as air-to-air tracking, formation flying, and approach. It does not apply to open-loop aircraft characteristics such as yaw response to a gust. Table 4.5 presents the Cooper-Harper rating scale. Aircraft controllability, pilot compensation (workload), and task performance are key factors in the pilot's evaluation. A Cooper-Harper rating of "one" is highest or best and a rating of "ten" is the worst, indicating the aircraft cannot be controlled during a portion of the task and that improvement is mandatory. Rating of one through three generally correspond to Level 1 flying qualities, a rating of four through six corresponds to level 2 flying qualities, and a rating of seven trough nine corresponds to Level 3.

Table 4.5: Cooper-Harper Scale

Pilot rating	Aircraft Characteristic	Demand of Pilot	Overall Assessment
1	Excellent, highly desirable	Pilot compensation not a factor for desired performance	Good flying Qualities
2	Good, negligible deficiencies	Pilot compensation not a factor for desired performance	
3	Fair, some mildly unpleasant deficiencies	Minimal pilot compensation required for desired performance	Flying qualities warrant improvement
4	Minor but annoying deficiencies	Desired performance requires moderate pilot compensation	
5	Moderately objectionable deficiencies	Adequate performance requires considerable pilot compensation	
6	Very objectionable but tolerable deficiencies	Adequate performance requires extensive pilot compensation	
7	Major deficiencies	Adequate performance not attainable with maximum tolerable pilot compensation; Controllability not in question	Flying quality deficiencies require improvement
8	Major deficiencies	Considerable pilot compensation is required for control	
9	Major deficiencies	Intense pilot compensation is required to retain control	
10	Major deficiencies	Control will be lost during some portion of required operation	Improvement Mandatory

4.6 Stability Augmentation System- displacement and rate feedback: Stability Augmentation Systems (SAS) were generally the first feedback control systems intended to improve dynamic stability characteristic. They were also referred to as dampers, stabilizers and stability augmenters. These systems generally fed back an aircraft motion parameter, such as pitch rate, to provide a control deflection that opposed the motion and increased damping characteristics. The SAS has to be integrated with primary flight control system of the aircraft consisting of the stick, pushrod, cables, and bell cranks leading to the control surface or the hydraulic actuator that activated the control surface. Fig 4.9 presents a simplified SAS. SAS sensors and computers are normally dual redundant to improve the reliability.

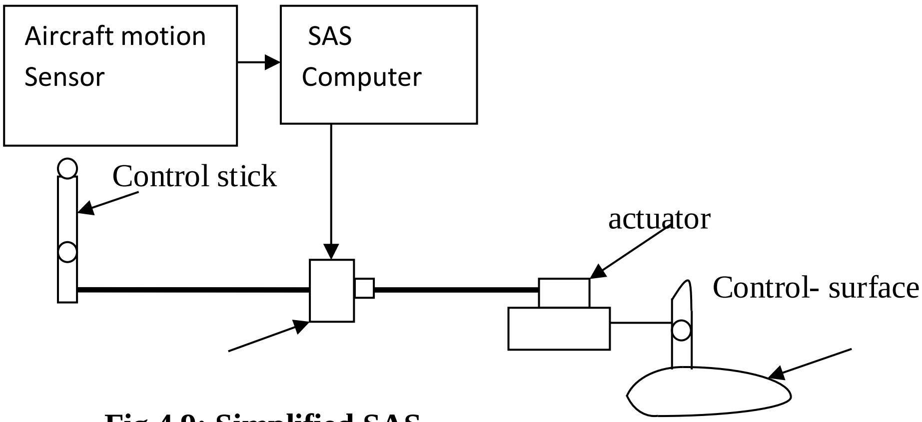


Fig 4.9: Simplified SAS

A closed loop system illustrating functions performed within a flight control computer is shown in fig 4.10.

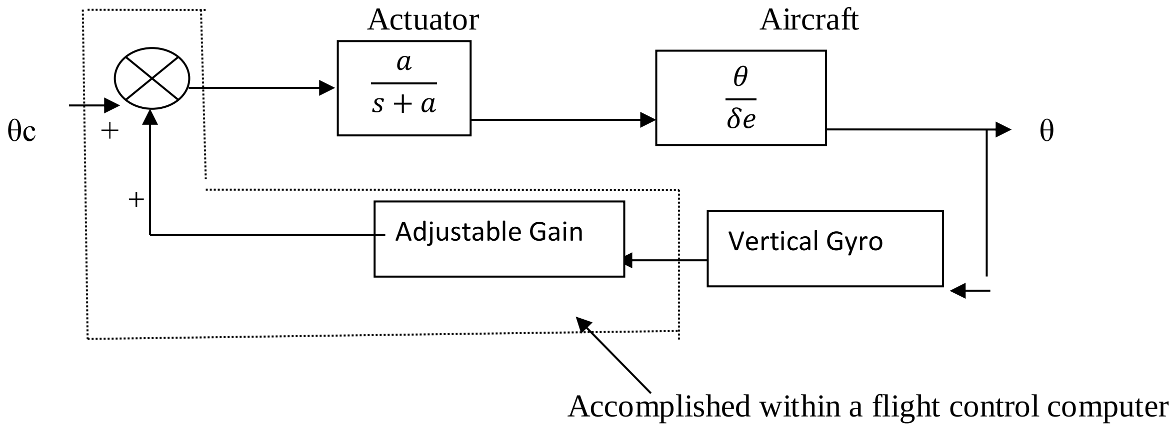


Fig 4.10 Closed loop system illustrating the functions performed within a flight control computer

The command signal and vertical gyro signal are input to the computer in the form of voltage or digital signals. Computer software multiplies the vertical gyro signal by the value of the adjustable gain (which is fixed for a final configuration), and then performs the comparator subtraction. Finally, the computer outputs the error signal (E) to an electromechanical actuator in the form of a voltage. The electro-mechanical actuator converts the voltage to a mechanical displacement, which is input into the control valve of the aircraft hydraulic actuator. Many aircraft integrate the electromechanical actuator with the hydraulic actuator as one unit.

4.6.1 Displacement (Position) feedback as a tool in SAS. A generalized transfer function (TF) of a second order system can be written as:

X(s)/Y(s) = $\frac{\omega_n{}^2}{s^2+2\xi\omega_n\, s+\omega_n{}^2}$; X(s) is output; Y(s) is input.

The TF may represent short-period mode of an aircraft with natural frequency and damping ratio representing the dynamic characteristic of the basic airframe. Fig 4.11 represents a simple closed loop position feedback system. The term "position" refers to the fact that output variable (x) is feedback as itself (not as derivative of x).

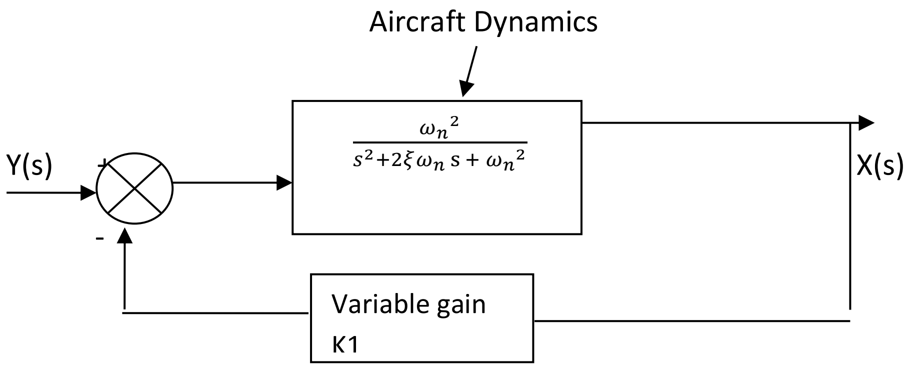


Figure 4.11: Position feedback system.

Closed loop transfer function of the position feedback system shown in fig 4.11 is:

X(s)/Y(s) = $\frac{\omega_n{}^2}{s^2+2\xi\omega_n\, s+\omega_n{}^2\,(1+K_1)}$

The closed loop characteristic equation of this system is

$s^2 + 2\xi\omega_n\, s + \omega_n{}^2\,(1+K_1) = 0$

From this equation we can see that

$\omega_{n\,position\;feedback} = \omega_n\,\sqrt{(1+K_1)}$ (1)

Likewise, closed- loop damping ratio has become

$\xi_{position\;feedback} = \frac{\xi}{\sqrt{(1+K_1)}}$ (2)

The important point is that the closed loop position feedback system provides the opportunity to change both the basic airframe natural frequency and damping ratio by adjusting the variable gain K_1. As can be seen from equation (1), position feedback allows the designer to increase the natural frequency of the closed loop system as K_1 is increased positively from zero. It is unfortunate that the closed-loop damping ratio (equation (2)) decreases as K_1 is increased. Closed-loop time constant (1/($\omega_{n\,position\;feedback}$ *$\xi_{position\;feedback}$) remains constant.

Thus, a **position feedback system provides the advantage** of automatic control of a motion variable (as in pitch hold example) but will be accompanied by an **increase in natural frequency**, a **decrease in damping ratio**, and **no change in time constant**.

4.7.2 Rate feedback System: Fig 4.12 presents a simple closed-loop rate feedback system. Rate refers to the fact that the derivative of the output variable (x) is feedback. Not that $\dot{x}$ (s) is generated by simply multiplying x(s) by the Laplace transform operator s.

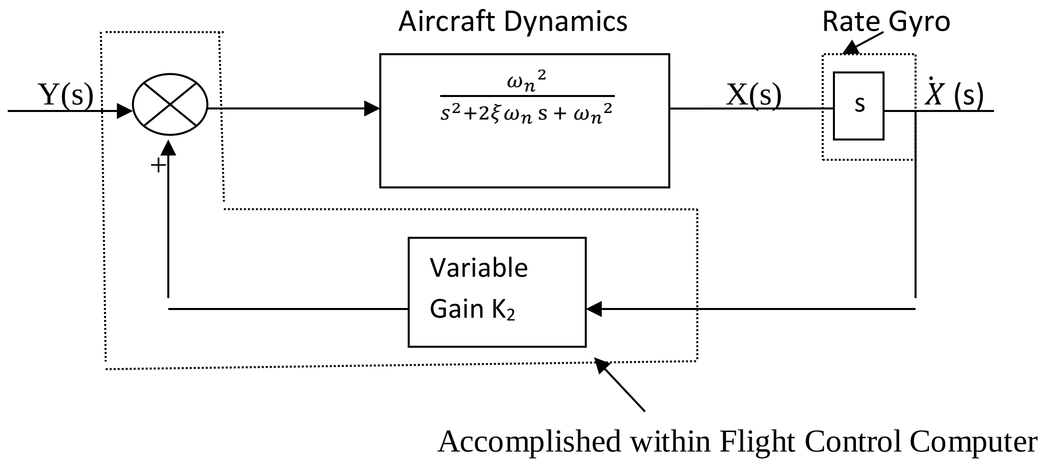


Fig 4.12: Rate feedback SAS

Closed-loop transfer function is

$$X(s)/Y(s) = \frac{\omega_n^2}{s^2+(2\xi\omega_n + K_2\,\omega_n^2)s+\omega_n^2}$$

The closed-loop characteristic equation for the system is

$$s^2 + (2\xi\omega_n + K_2\,\omega_n^2)s + \omega_n^2 = 0$$

We can see from the above characteristic equation that the natural frequency of the open-loop &closed-loop system remains constant and is not affected by the value of K2. The closed loop damping ratio becomes

$$\xi_{rate\ feedback} = \frac{2\xi+K_2\omega_n}{2}$$

Rate feedback allows the designer to increase the damping ratio as K2 is increased positively from zero. This provides a powerful design tool to tailor the handling qualities of an aircraft and meet dynamic stability damping ratio requirements. A rate feedback system typically involves adding a rate gyro to the aircraft to provide $\dot{x}$ measurement and feedback signal shown in fig 4.12. The figure illustrates where the rate gyro fits into the system. A rate gyro is a sensor that outputs a voltage proportional to an angular rate. Most highly augmented aircraft have pitch rate (Q), roll rate (P), and yaw rate (R) gyros to tailor dynamic stability and response characteristics for all three rotational degrees of freedom.

4.7.3 Acceleration feedback: Fig 4.13 below shows a simple closed-loop acceleration feedback system. Acceleration refers to the fact that the second derivative of the output variable (x) is feedback.

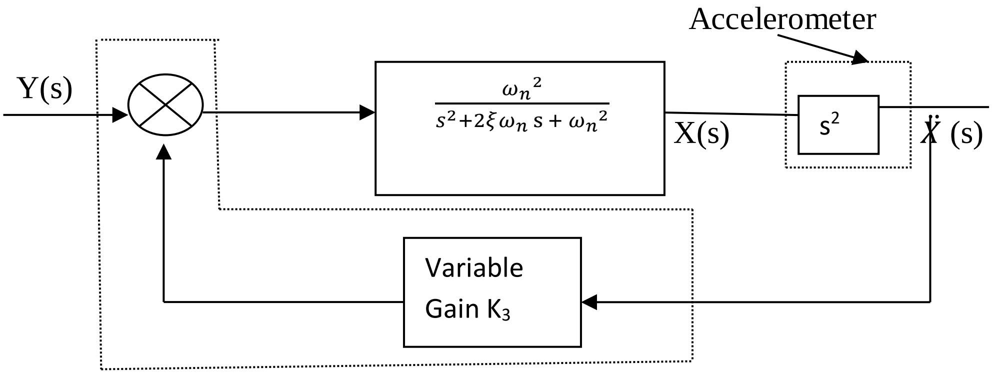


Fig 4.13: Acceleration feedback

Closed-loop transfer function is

$$X(s)/Y(s) = \frac{\omega_n^2}{(1+K_3\,\omega_n^2)s^2+2\xi\omega_n\,s+\omega_n^2}$$

The closed loop characteristic equation for the system is

$$s^2+\frac{2\xi\omega_n}{1+K_3\,\omega_n^2}\,s+\frac{\omega_n^2}{K_3\,\omega_n^2}=0$$

The closed loop frequency becomes

$$\omega_{n\,\substack{acceleration\\feedback}}=\frac{\omega_n}{\sqrt{1+K_3\,\omega_n^2}}$$

Closed loop damping ratio $\xi_{\substack{\boldsymbol{acceleration}\\\boldsymbol{feedback}}}=\frac{\xi}{\sqrt{1+K_3\,\omega_n^2}}$

These equations indicate that natural frequency & damping ratio are decreased as K3 is increased positively from zero. With position, rate & acceleration feedback, we have the ability to increase or decrease the natural frequency & damping ratio of an open loop system. Handling qualities of an aircraft can be tailored with these tools & the roots of the characteristic equation can be positioned in the complex plane to meet stated requirements. In some cases, a combination of position, rate, and/or acceleration feedback is needed to achieve the desired characteristic. A multi loop system using all three types of system is shown in fig 4.14.

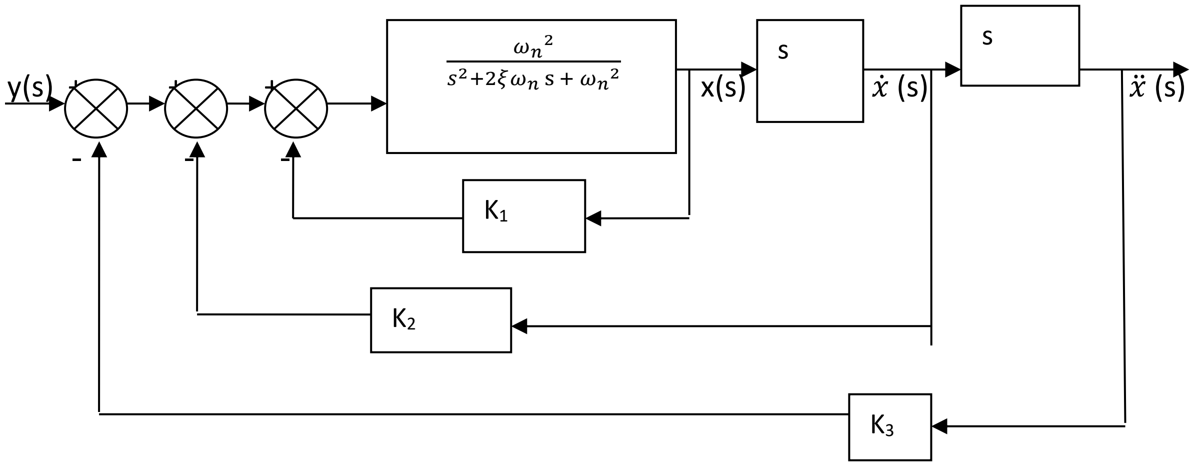


Fig 4.14: A multi loop system using tree feedback loops.

4.8 Control Augmentation system: Control augmentation system (CAS) added a pilot command input into the flight control computer. A force sensor on the control stick was usually used to provide this command input. With CAS, a pilot stick input is provided to FCS in two ways- through the mechanical system and through the CAS electrical path. The CAS design eliminated the SAS problem of pilot inputs being opposed by the feedback. Fig 4.15 presents a simplified CAS.

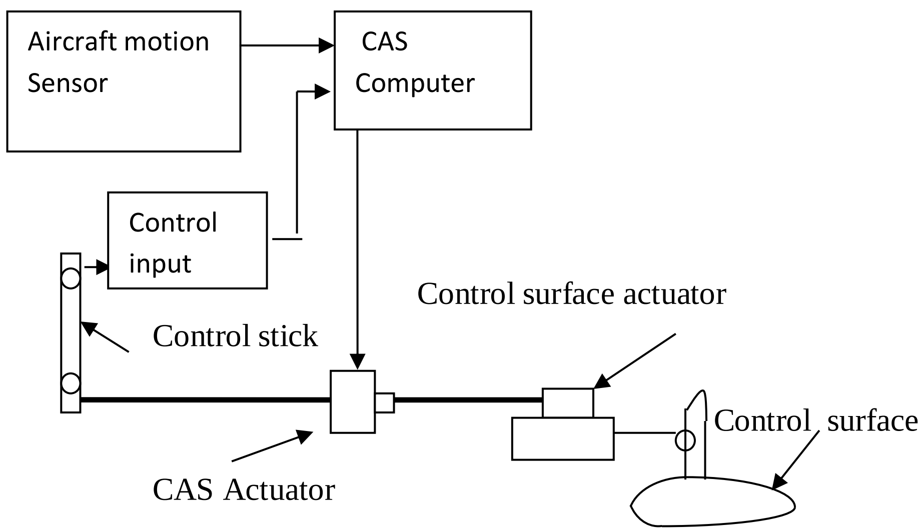


Fig 4.15: Simplified CAS

With CAS, aircraft dynamic response is typically well-damped, and control response is scheduled with the control system gains to maintain desirable characteristic throughout the flight envelop. A block diagram of a typical CAS is presented in fig 4.16.

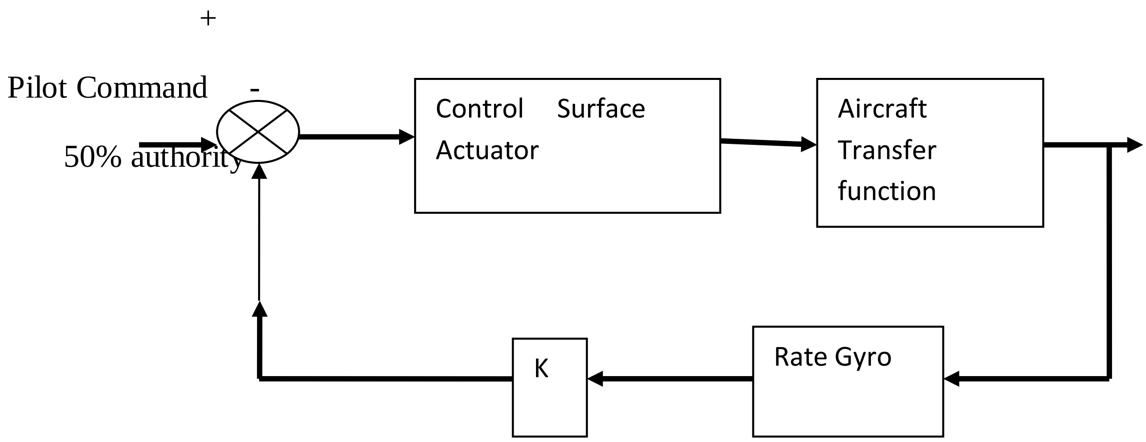


Fig 4.16: Simplified diagram of CAS

CAS provides dramatic improvement in aircraft handling qualities. Both dynamic stability and control response characteristics could be tailored and optimized for the mission of the aircraft. In case of high-performance military aircraft, where the pilot may have to maneuver the aircraft to its performance limits and perform tasks such as precision tracking of targets, specialized CAS are needed. FCS can provide the pilot with selectable "task tailored control laws". For example, although the role of a fighter aircraft has changed to include launching missiles from long range, the importance of the classical dogfight is still recognized. A dogfight places a premium on high maneuverability and "agility" (ability to maneuver quickly) in the aircraft and control system that allows the pilot to take advantage of this maneuverability. In this situation a suitable controlled variable for pitch axis is the normal acceleration of the aircraft. This is the component of acceleration in the negative direction of body-fixed z-axis. It is directly relevant to performing a maximum-rate turn and must be controlled up to the structural limits of the airframe, or the pilot's physical limits. Therefore, for a dogfight, a "g-command" control system is an appropriate mode of operation of the FCS. Another common mode of operation for a pitch-axis CAS system is a pitch rate command system. When a mission requires precise tracking of a target, by means of a sighting device, it has been found that a deadbeat response to pitch-rate commands is well suited to the task. Control of pitch rate is also the preferred system for approach and landing. A pitch rate CAS is shown in the fig 4.17.

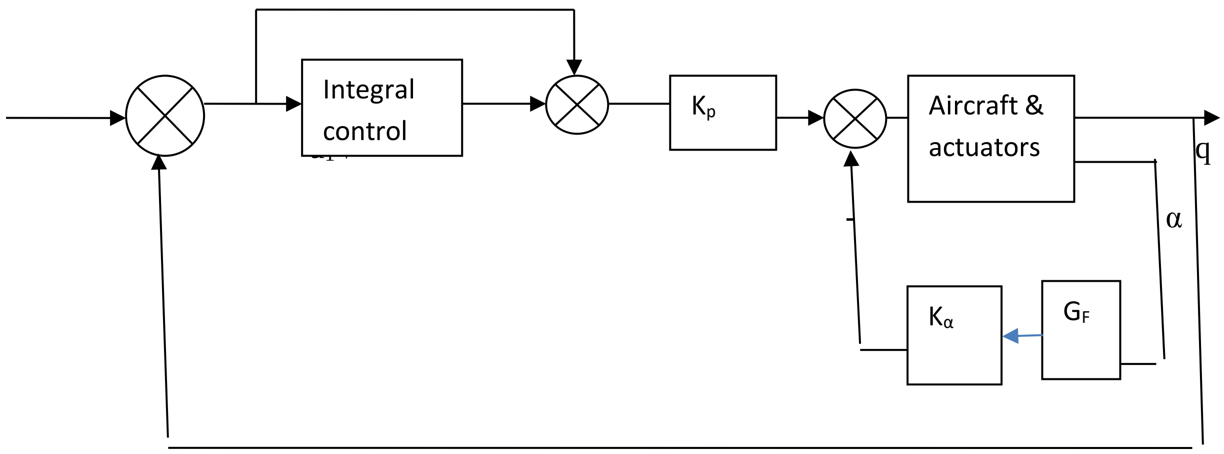


Fig 4.17: Pitch-rate control-augmentation

A normal-acceleration control augmentation system is shown in fig 4.18.

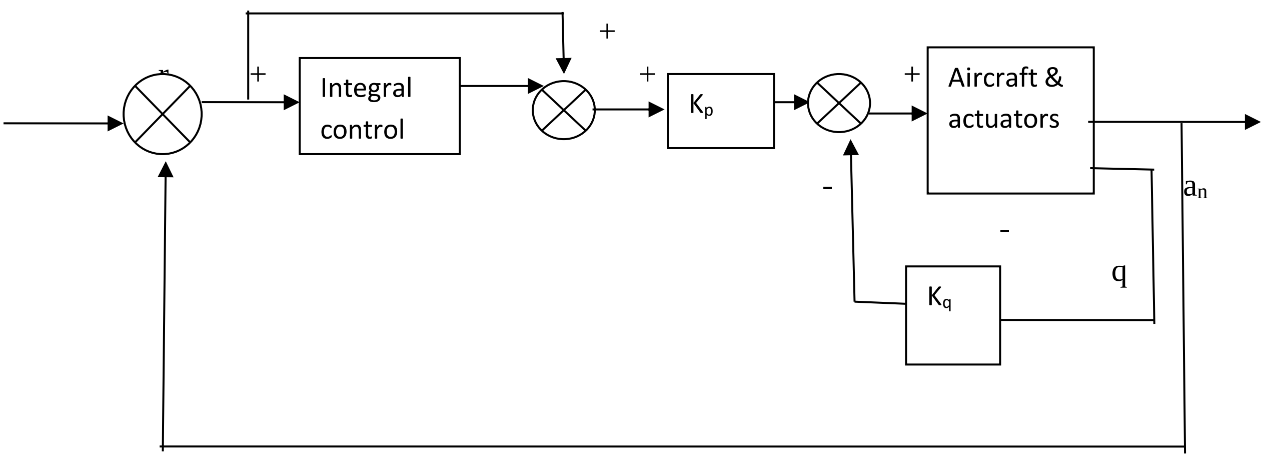


Fig 4.18: Normal Acceleration CAS

4.9 Full authority fly-by-wire control: Full authority fly-by-wire (FBW) system has no mechanical link from the control stick to actuator system. Basically, FBW systems are CAS system without mechanical control system and provide the CAS full authority. The input from control stick, pedal and from motion sensors are converted into electrical signals and sent to FBW computer. Software inside the FBW computer contains the control law which will command the control surfaces to move. However, to improve the reliability, triple and quad redundancy in system components along with self-test software is used. Aircraft such as F-16, Mirage-2000 and Tejas have FBW FCS. The full authority provided by FBW allows significant tailoring of stability and control characteristics. This ability has led to FBW systems with several feedback parameters and weighting of feedback gains based on flight condition and other parameter. Fig 4.19 presents a simplified FBW system.

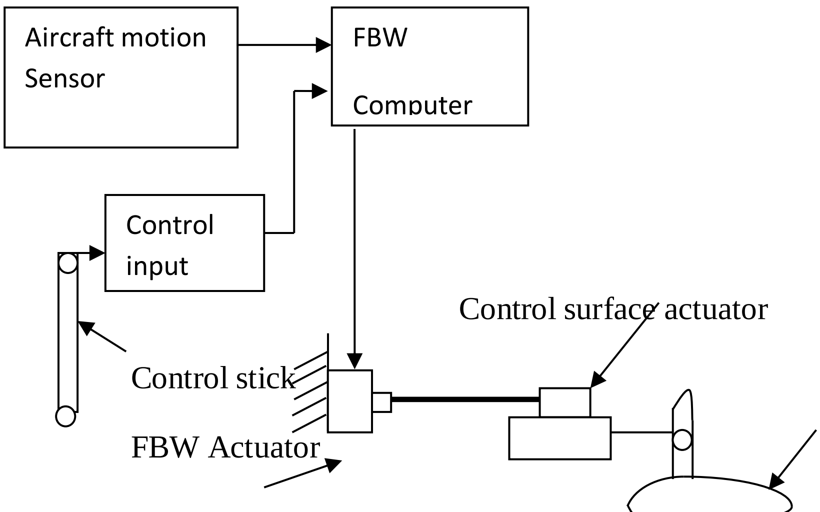


Fig 4.19: Simplified FBW

Block diagram of F-16 longitudinal FBW system is shown in fig 4.20.

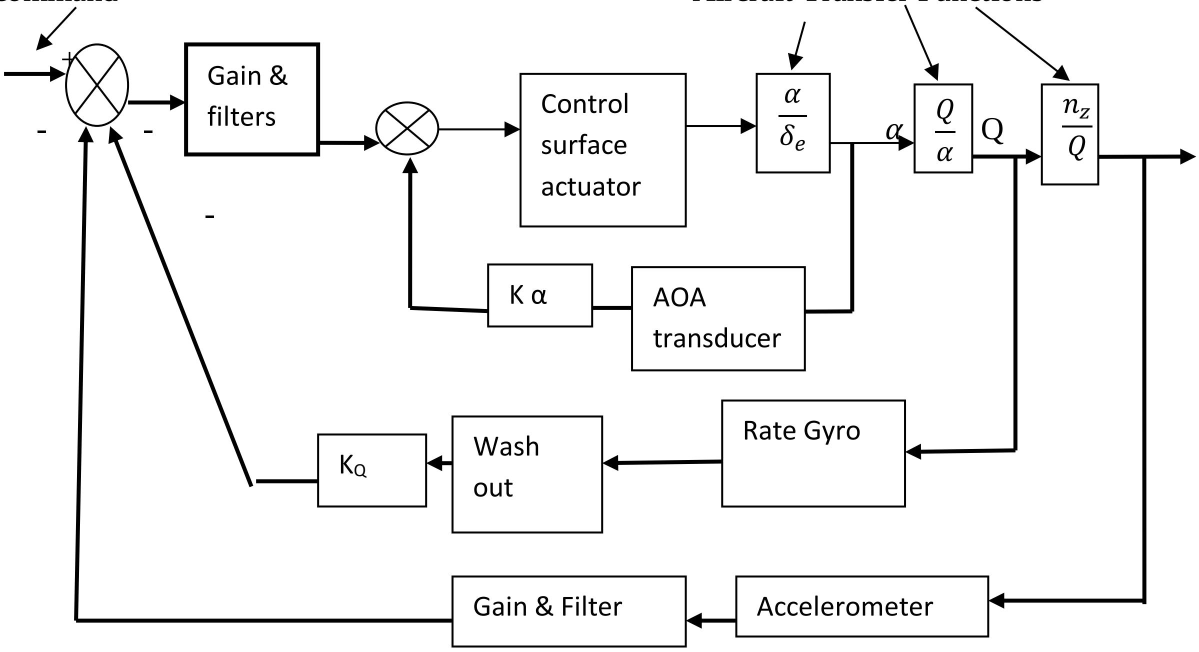


Fig 4.20: Simplified F-16 longitudinal FBW block diagram

Advantages of FBW Control:

(a) Increased performance: FBW enables a smaller tail plane, fin rudder to be used, thereby reducing both aircraft weight and drag, active control of the airplane and rudder making up for the reduction in natural stability. For a civil airliner, reducing the stability margin and compensating for the reduction with a FBW system thus results in lighter aircraft with better performance and better operating economics and flexibility than a conventional design, for example, the ability to carry additional freight. For a military aircraft, such as an air superiority fighter, the FBW system enables aircraft configurations with negative stability to be used. This gives more lift, as the trim lift is positive, so that a lighter, more agile fighter can be produced- agility defined as the ability to change the direction of the aircraft's velocity vector. An increase in instantaneous turn rate of 35% is claimed for some of the new agile fighters.

(b) Reduced weight. Electrically signaled controls are lighter than mechanically signaled controls. FBW eliminates the bulk and mechanical complexity of mechanically signaled controls with their disadvantages of friction, back lash (mechanical lost motion), structure flexure problem, periodic rigging and adjustments.

(c) FBW control stick: FBW flight control enables a small, compact pilot's control stick to be used allowing more flexibility in cockpit layout. The displays are un-obscured.

(d) Automatic stabilization.

(e) Carefree Maneuvering. The FBW computer continuously monitors the aircraft's state to assess how close it is to its maneuver boundaries. It automatically limit's the pilot's command inputs to ensure that the aircraft does not enter an unacceptable attitude or approach too near its limiting incidence angle (approaching the stall) or carry out maneuver which would exceed the structure limits of the aircraft. A number of aircraft are lost each year due to flying too close to their maneuver limits and the very high workload in the event of a subsequent emergency. The FBW system can thus make a significant contribution to flight safety.

(f) Ability to integrate additional controls. These controls need to be integrated automatically to avoid an excessive pilot-work-too many things to do at once:

(i) Leading and trailing edge flaps for maneuvering and not just for take-off and landing

(ii) Variable wing sweep

(iii) Thrust vectoring

(g) Ease of integration of the autopilot. The electrical interface and the maneuver command control of the FBW system greatly ease the autopilot integration task. The autopilot provides steering commands as pitch rate or roll rate commands to the FBW system. The relatively high bandwidth maneuver command 'inner loop' FBW system ensures that response to the outer loop autopilot commands is fast and well damped, ensuring good control of the aircraft flight path in the autopilot modes. A demanding autopilot mode performance is required for applications such as automatic landing, or, automatic terrain following at 100-200 ft above the ground at over 600 knots where the excursions from the demanded flight path must be kept small.

(h) Aerodynamics versus 'Stealth': The concept of reducing the radar cross-section of an aircraft so that it is virtually undetectable has been given the name 'stealth' in the USA. Radar reflection returns are minimized by faceted surfaces which reflect radar energy away from the direction of the source, engine intake design and the extensive use of radar energy absorbing materials in the structure. Stealth considerations and requirements can conflict with aerodynamics requirements and FBW flight control is essential to give acceptable, safe handling across the flight envelop. The feedback must be adjusted according to flight condition. The adjustment process is called gain scheduling because, in its simplest form, it involves only changing the amount of feedback as a function of scheduling variable. These scheduling variables will normally be measured dynamic pressure and/or Mach number. The signals from rate gyros, accelerometers, air data computer, and other sources are processed by the flight-control computer (FCC).

4.10 Need for automatic Control: Fig 4.21 shows the altitude-Mach envelope of a modern high-performance aircraft; the boundaries of this envelop are determined by a number of factors. The low-speed limit is set by the maximum lift that can be generated (the alpha limit in the figure), and the high-speed limit follows a constant dynamic pressure contour (because of structural limits, including temperature). At high altitudes the speed becomes limited by the maximum engine thrust (which falls off with altitude). The altitude limit imposed on the envelop is where the combination of airframe and engine characteristics can no longer produce a minimum rate of climb (this is the "service ceiling"). The basic aerodynamic coefficients (stability derivatives) vary with Mach number. Because of the large changes in aircraft dynamics, a dynamic mode that is stable and adequately damped in one flight condition may become unstable, or at least inadequately damped, in another flight condition. A lightly damped oscillatory mode may cause a great deal of discomfort to passengers or make it difficult for the pilot to control the trajectory precisely. These problems are overcome by using feedback control to modify the aircraft dynamics. The aircraft motion variables are sensed and used to generate signals that can be fed into the aircraft control-surface actuators, thus modifying the dynamic electrical output of the FCC is used to drive electro hydraulic valves, and these superimpose additional motion on the hydro mechanical control system.

One may ask as to why use an FCC instead of pilot? There are several reasons for this. First of all, a computer has a much higher reaction velocity than a pilot. Also, it isn't subject to concentration losses and fatigue. Finally, a computer can more accurately know the state of the aircraft is in. (Computer can handle huge amount of data better and also don't need to read a small indicator to know, for example, the velocity or the height of the aircraft.)

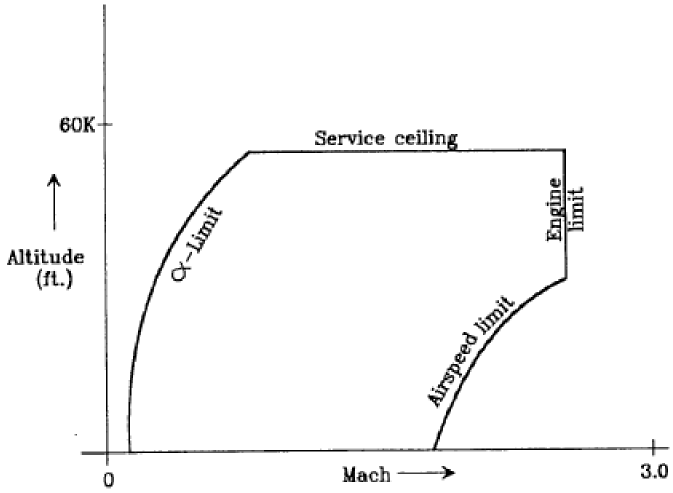


Fig 4.21 Aircraft altitude-Mach envelope

Fig 4.22 shows how a fully powered aircraft control system might be implemented with mechanical, hydraulic, and electrical components.

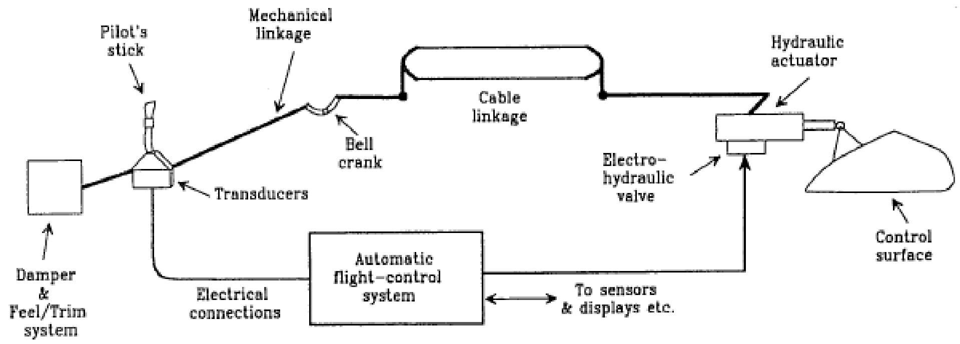


Fig 4.22: An electro-mechanical control system

4.11. Autopilots-purpose, functioning-inputs-hold-command-track.

4.11.1 Autopilots-Purpose and Functioning, inputs-hold, command, track: Basic purpose of an auto-pilot is to reduce the pilot work load (pilot- relief auto-pilot). The auto-pilots are capable of maintaining (holding) constant attitude (pitch, roll, and heading), velocity, and altitude. They can also be coupled to instrument landing system during landing in bad weather conditions. In automatic terrain following mode they can be used to fly in a hilly terrain without much work load on the pilot. They can also be used as SAS. Auto-pilots are used for tracking a command instead of holding a reference value. In such cases reference command may be pitch-rate or normal acceleration. Maneuvering auto-pilots can be used in high performance fighter aircraft to give desired normal acceleration, turn rate and pitch-rate during various modes of combat (example dogfight, air-to-ground target tracking). In hold autopilot a constant output is maintained like in heading hold mode present heading is maintained once the heading hold mode of the auto-pilot is engaged. In command input, auto-pilot is commanded (e.g. a new given heading or bank angle) to new state (bank angel, altitude, heading).

4.12 Displacement Autopilots-Pitch, yaw, bank, altitude and velocity hold-purpose, relevant simplified aircraft transfer functions, feedback signals:

4.12.1 Displacement autopilot-pitch, yaw autopilot: One of the earliest auto-pilots to be used for aircraft control is the so-called displacement auto-pilot. A displacement type autopilot can be used to control the angular orientation of the airplane. Conceptually, the displacement autopilot works in the following manner. In a pitch attitude displacement autopilot, the pitch angle is sensed by a vertical gyro and compared with the desired pitch angle to create an error angle. The difference or error in pitch attitude is used to produce proportional displacements of the elevator so that the error signal is reduced. Figure 4.23 is a block diagram of either a pitch or roll angle displacement autopilot. The heading angle of the airplane also can be controlled in a similar scheme. The heading angle is sensed by a directional gyro and the error signal is used to displace the rudder to reduce the error signal. A displacement heading auto pilot is shown in fig 4.24.

4.12.2. Bank Attitude autopilot. The roll attitude of an airplane can be controlled by a simple bank angle autopilot as illustrated in fig 4.25. Conceptually the roll angle of the airplane can be maintained at whatever angle one desires. IN practice we would typically design the autopilot to maintain a wings level attitude or $\varphi = 0$. The autopilot is composed of a comparator, aileron actuator, aircraft equation of motion (i.e. transfer function), and an attitude gyro to measure the airplane's roll angle.

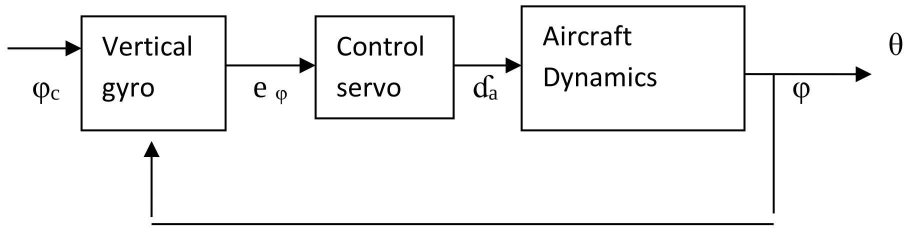


Fig 4.23: A roll or pitch displacement autopilot

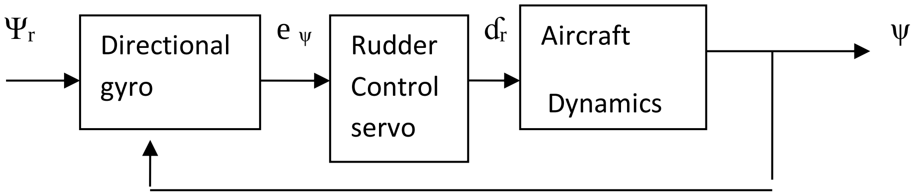


Fig 4.24: A heading displacement autopilot

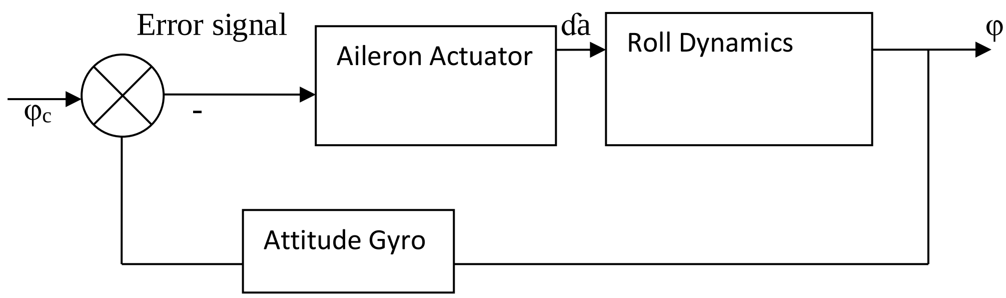


Fig 4.25: Simple roll attitude control system

4.12.3 Altitude hold autopilot: The altitude of an airplane can be maintained by an altitude hold autopilot. A simple altitude hold autopilot is shown in fig 4.26. Basically, the autopilot is constructed to minimize the deviation between the actual altitude and the desired altitude. To analyze how such an autopilot would function we examine an idealized case. First, we assume that the airplane's speed will be controlled by a separate control system, second, we neglect any lateral dynamics. With these restrictions we are assuming that the only motion possible is in vertical plane. The transfer functions necessary to perform the analysis are elevator servo and aircraft dynamics. The elevator transfer function can be represented as a first order lag as $\frac{\delta_e}{e} = \frac{k_a}{s+10}$. The aircraft dynamics can be represented by short period approximations. Next, we need to find the transfer function $\Delta h/\Delta\delta e$. This can be shown as

$$\frac{\Delta h(s)}{\Delta\delta_e\,(s)} = \frac{u_0}{s}\left[\frac{\Delta\theta(s)}{\Delta\delta_e(s)} - \frac{\Delta\alpha(s)}{\Delta\delta_e(s)}\right]$$

The transfer function $\frac{\Delta\theta(s)}{\Delta\delta_e(s)}$ can be obtained from $\Delta q(s)/\Delta\delta e(s)$ in the following ways

$\Delta q = \Delta\dot{\theta}$; hence Δq(s) = s $\Delta\theta(s)$

Hence $\frac{\Delta\theta(s)}{\Delta\delta_e(s)} = \frac{1}{s}\left(\Delta q(s)/\Delta\delta e(s)\right) = \frac{A_q\ s+B_q}{s(As^2+Bs+C)}$

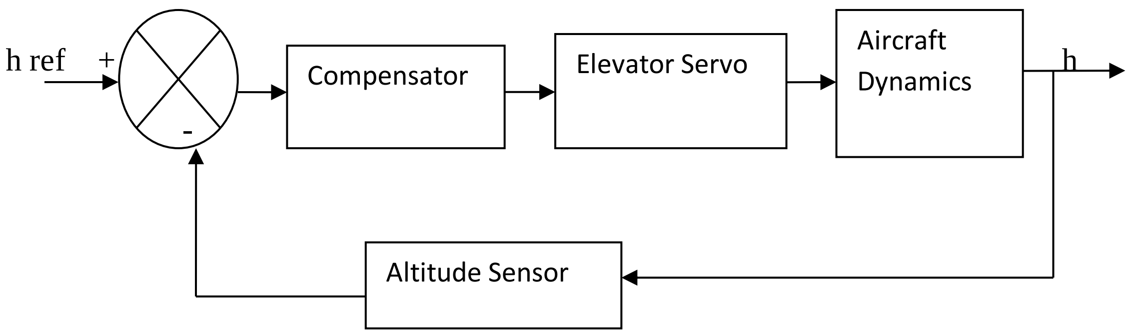


Fig 4.26: Altitude hold autopilot

4.12.4. Velocity Hold Autopilot: The forward speed of an airplane can be controlled by changing the thrust produced by the propulsion system. The function of the speed control system is to maintain some desired flight speed. This is accomplished by changing the engine throttle setting to increase or decrease the engine thrust. Figure 4.27 is simplified concept for a speed control system. The components that make up the system include a compensator, engine throttle, aircraft dynamics, and feedback path consisting of the velocity and acceleration feedback.

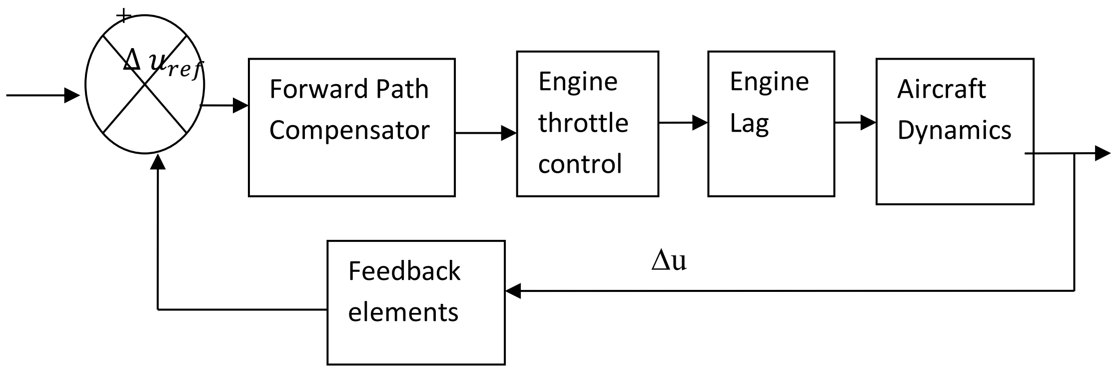


Figure 4.27: A block diagram for a speed control system

4.13 Autopilot design by displacement & rate feedback-iterative methods, design by displacement feedback and series PID compensator-Zeigler & Nichols method:

4.13.1 Design of autopilot by displacement& rate feedback using iterative methods: Design of an autopilot by displacement & rate feedback is explained with an example of a pitch attitude hold auto pilot of a transport aircraft. Basic block diagram of a pitch hold auto pilot is shown in fig 4.33. For this design reference the reference pitch angle is compared with the actual pith angle measured by the pitch gyro to produce an error signal to activate the control surface actuator to deflect the control surface. Movement of the control surface causes the aircraft to achieve a new pitch orientation, which is feedback to close the loop. To design the control system for this auto pilot we need the transfer function of each component. The transfer function of the elevator servo can be represented as a first order system

θe/v $=k_a/(\tau s+1)$; where δe,v, k_a and τ are the elevator deflection angle, input voltage,elevator servo gain and servo motor time constant. The time constant can be assumed to be 0.1s. we can represent the aircraft dynamics by short-period approximation. The short period TF for the a typical jet transport for example can be written as:

$$\frac{\Delta\theta}{\Delta\delta e} = -2.0(s + 0.3)/(s(s^2+0.65s+2.15))$$

The fig 4.28 is the block diagram representation of the auto pilot.The problem now is one of determing amplifier gain k_a so that the control system will have the desired performance. Selection of k_a can be determined using a root locus plot of transfer function. Fig 4.29 is the root locus plot for the typical jet aircraft pitch control autopilot. As the gain is increased from zero, the system damping decreases rapidly and the system becomes unstable. Even for low values k_a, the system damping would be too low for satisfactory dynamic performance. The reason for poor performance is that the airplane has very little natural damping. To improve the design we could increase the damping of the short period mode by adding an inner loop feedback loop. Fig 4.30 is a block diagram representing of displacement auto pilot with pitch rate feed back for imprved damping. In the inner loop the pitch rate is measured by a rate gyro and fed back to be added with error signal generated by the difference in pitch attitude. Fig 4.31 shows the block diagram for the business jet where pitch rate is incorporated into the design. For this problem we have two parameters to select, namely the gains k_a and k_{rg}. The root locus method can be used to pick both parameters. The procedure is essentially a trial-and-error method. First,the root locus diagram is determined for the inner loop; a gyro gain is selected, and then the outer root locus plot is constructed. Several iterations may be required until the desired overall system performance is achieved.

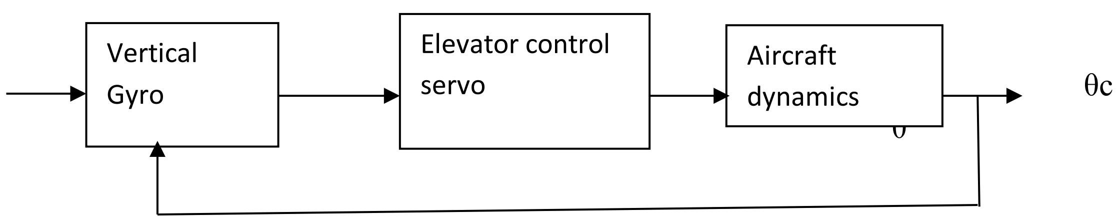


Fig 4.28 A pitch displacement autopilot

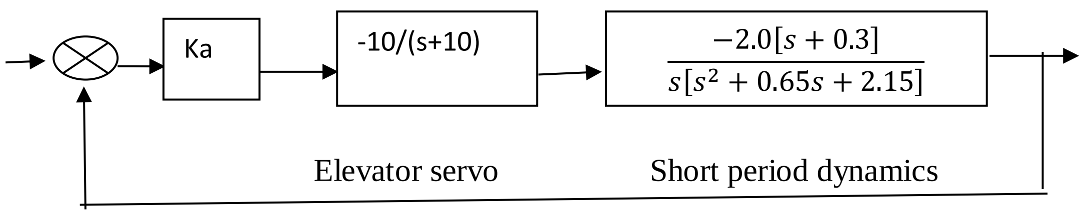


Fig 4.29 A pitch displacement autopilot

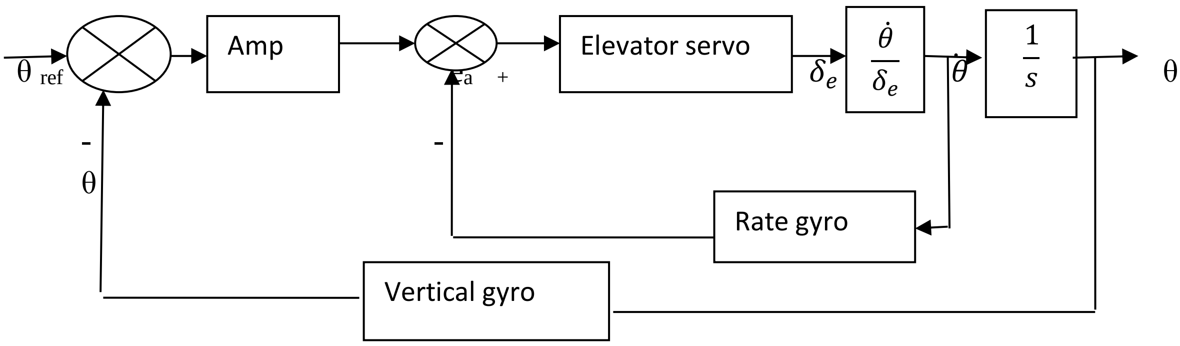


Fig 4.30: A pitch attitude autopilot employing pitch rate feedback

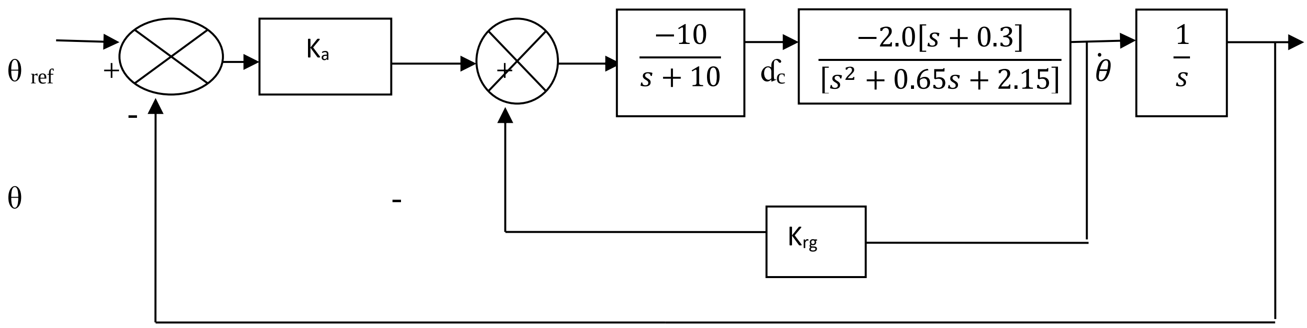


Fig 4.31: A pitch attitude autopilot employing pitch rate feedback

4.13.2 PID controller or Ziegler-Nichols tuning rules: The simplest feedback controller is one for which the controller output is proportional to the error signal. Such a controller is called a proportional to the error signal. Such a controller is called a proportional control. Obviously. the controller's main advantage is its simplicity. It has the disadvantage that there may be a steady state error. The steady-state error can be eliminated by using an integral controller

$e_o(s) = k_i \int_0^t e(t)dt$ or $e_o(s) = k_i \frac{e(s)}{s}$ where k_i is the integral gain. The advantage of the integral controller is that the output is proportional to the accumulated error. The disadvantage of the integral controller is that we make the system less stable by adding the pole at the origin. Recall that the addition of a pole to the forward- path transfer function is to bend the root locus toward the right half s-plane. It is also possible to use a derivative controller defined as is that the controller:

$e_o(t) = k_d \frac{de}{dt}$ or $e_o(s) = k_d s(s)$

The advantage of the derivative controller is that the controller will provide large corrections before the error becomes large. The major disadvantage of the derivative controller is that it will not produce a control output if the error is constant. Another difficulty of the derivative controller is its susceptibility to noise. The derivative controller in its present form would have difficulty with noise problem. This can be avoided by using a derivative controller of the form

$e_o(s) = k_d \frac{s}{\tau s+1} e(s)$

The term $1/(\tau s + 1)$ attenuates the high- frequency components in the error signal, that is, noise, thus avoiding the noise problem. Each of the controller-providing proportional, integral, and derivative control-has its advantages and disadvantages. The disadvantages of each controller can be eliminated by combining all three controllers into a single PID controller, or proportional, integral, and derivative, controller.

The selection of the gains for the PID controller can be determined by a method developed by Ziegler and Nichols, who studied the performance of PID controllers by examining the integral of the absolute error (IAE):

$\text{IAE} = \int_0^\infty |e(t)|\, dt$

From their analysis they observed that when the error index was a minimum the control system responded to a step input as shown in fig 4.32. Note that second overshoot is one quarter of the magnitude of the maximum overshoot. Based on their analysis they derived a set of rules for selecting the PID gains. The gains k_p, k_i, and k_d are determined in terms of two parameters, k_{pu}, called the ultimate gain, and T_u, the period of oscillation that occurs at the ultimate gain. Table 4.9 gives the values for the gains for proportional (P), proportional-integral (PI), and the proportional-integral-derivative (PID) controllers.

To apply this technique the root locus plot for the control system with the integral and derivative gains set to 0 must become marginally stable. That is, as proportional gain is increased the locus must intersect the imaginary axis. The proportional gain, k_p , for which this occurs is called the ultimate gain, k_{pu}. The purely imaginary roots, $\lambda = \pm j\omega$, determine the value of T_u .

$Tu = \frac{2\pi}{\omega}$

One additional restriction must be met: All other roots of the system must have negative real parts; that is, they must be in the left-hand portion of the complex s-plane. If these restrictions are satisfied the P, PI, or PID gains easily can be determined.

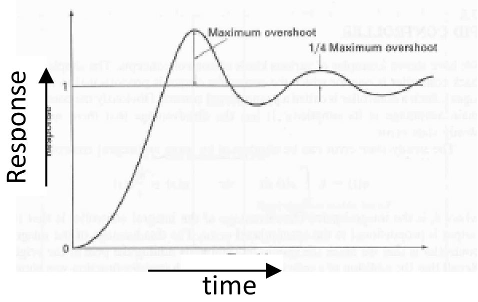


Fig 4.32: The quarter decay

Table 4.9 Gains for P, PI, and PID Controllers

Type of controller	$\mathbf{k_p}$	$\mathbf{k_i}$	$\mathbf{k_d}$
P(proportional controller)	$k_p = 0.5\ k_{pu}$		
PI (proportional-integral controller)	$k_p = 0.45\ k_{pu}$	$k_i = 0.45\ k_{pu}/(0.83\ T_u)$	
PID(Proportional-integral-derivative controller)	$k_p = 0.6\ k_{pu}$	$k_i = 0.6\ k_{pu}/(0.5\ T_u)$	$k_d = 0.6\ k_{pu}\ (0.125\ T_u)$

Example Problem: Design a PID controller for the controller for the control system shown in fig 4.33.

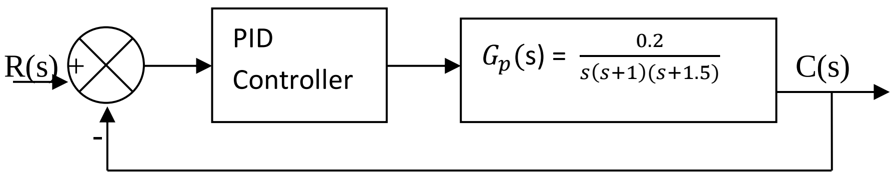


Fig 4.33 PID controller

Solution: The gains of the PID controller can be estimated using the Ziegler-Nichols method provided the root locus for the plant becomes marginally stable for some value of the proportional gain k_p when the integral and derivative control gains have been set to 0. The root locus plot for G(s) $=\frac{0.2k_p}{s(s+1)(s+1.5)}$

is shown in fig 4.34.The root locus plot meets the requirements for the Ziegler-Nichols method. Two branches of the root locus cross the imaginary axis and all other roots lie in the left half plane. The ultimate gain k_u is found by finding the gain when the root locus intersects the imaginary axis. The locus intersects the imaginary axis at s= $\pm$1.25j. The gain crossover point can be determined from the magnitude criteria:

$$\frac{|0.2k_p|}{|s||s+1|s+1.5||} = 1$$

Substituting s = 1.25j into the magnitude criteria yields

$k_{pu} = 19.8$

The period of un-damped oscillation Tu is obtained as follows:

$T_u = \frac{2\pi}{\omega} = \frac{2\pi}{1.25} = 5.03$

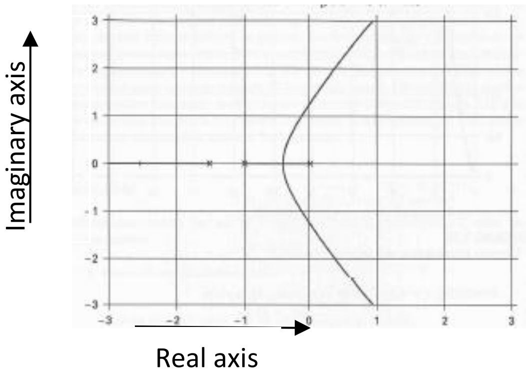
Knowing k_{pu} and T_u the proportional, integral, and derivative gains k_p, k_i , and k_d can be evaluated:

$k_p = 0.6\ k_{pu} = (0.6)(19.8) = 11.88$

$k_i = 0.6\ k_{pu} /(0.5\ T_u) = (0.6)(19.8)/[(0.5)(5.03)] = 4.73$

$k_d = 0.6\ k_{pu}\ (0.125T_u) = (0.6)(19.8)(0.125)(5.03) = 7.47$

The response of the control system to a step input is given in fig 4.35.



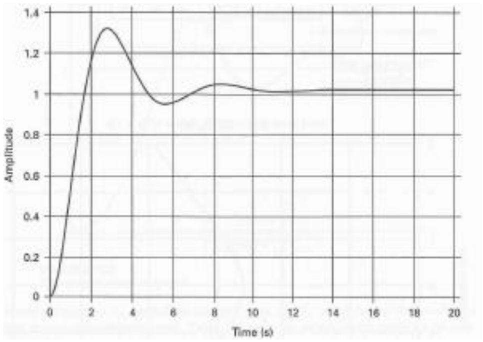


Fig 4.34 Root locus plot for G(s) =$\frac{0.2k_p}{s(s+1)(s+1.5)}$ **Fig 4.35 Transient response to a step input**

UNIT V

Modern Control Theory-State Space Modeling, Analysis:

Index

5.1 Limitations of Classical methods of control modeling, analysis and design, applied to complex, MIMO system:

(a) Transfer function models are used for linear time invariant (LTI) continuous time systems. These are called frequency response models due mainly to the interpretation of the Laplace transform variables s as complex frequency in contrast with differential equation models, which are time-domain models. Transfer function model has limitations as it cannot be applied to non-linear or linear time varying system. Furthermore, these models cannot be used efficiently for systems of higher orders or multi variable system (MIMO). Time-domain models or state space models are especially suitable for use with computers. These models can be used to study the non-linear or time varying system. Another important feature of the state space representation is that it gives information about the internal behavior of the system, as well as the input-output behavior of the system.

(b) In classical control design of feedback control is accomplished using the root locus technique and Bode methods. These techniques are very useful in designing many practical control problems. However, design of control system using root locus or Bode technique is trial & error procedure. The major advantage of these techniques is their simplicity & ease of use. The advantage disappears quickly as complexity of the system increases.

(c) With rapid development of high speed computers during the recent decade, a new approach to control system design has evolved. This new approach is called modern control theory. This theory permits a more systematic approach to control system design. In modern control theory, the control system is specified as a system of first-order differential equations. By formulating the problem in this manner, the control designer can fully exploit the digital computer for solving complex control problem. Another advantage of the modern control theory is that optimization techniques can be applied to design optimal control systems.

5.2 State space modeling of dynamical systems-state variable definition-state equations, the output variable-the output equation-representation by vector matrix first order differential equations:

5.2.1 State space modeling of dynamical system: The state space approach to control system design is a time domain method. The application of state variable technique to control problem is called modern control theory. The state equations are simply firs-order differential equations that govern the dynamics of the system being analyzed. It should be noted that any high order system can be decomposed into a set of first-order differential equation.

In mathematical sense, state variables and state equations completely describe the system.

Definition of State Variable: The state variable of a system are a minimum set of variables $x1(t)$, $x2(t)$…$xn(t)$ which, when known at time $t0$ and along with the input, are sufficient to determine the state of a system at any time $t > t0$.

Modeling of Dynamical systems, State Equations, the output variable, the output equations: Once a physical system has been reduced to a set of differential equations, the equation can be written in a convenient matrix form as:

$\dot{x}$ = A x + B η (1)

The output of the system is expressed in terms of state & control inputs as follows:

y = C x + D η (2)

The state, control, & output vectors are defined as follows:

x = $\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$; **State vector** n×1

η = $\begin{bmatrix} \delta_1(t) \\ \delta_2(t) \\ \vdots \\ \delta_p(t) \end{bmatrix}$; **Control or input vector** p × 1

y= $\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_q(t) \end{bmatrix}$; **output Vector** q× 1.

The matrix A, B, C, D are defined in the following manner

A = $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$; Plant matrix

B= $\begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}$; Control or input matrix n× p

C = $\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{q1} & \cdots & c_{qn} \end{bmatrix}$; q× n matrix

D = $\begin{bmatrix} d_{11} & \cdots & d_{1p} \\ \vdots & \ddots & \vdots \\ d_{q1} & \cdots & d_{qp} \end{bmatrix}$; q× p matrix

Fig 5.2.1 is a sketch of the block diagram representation of the state equation.

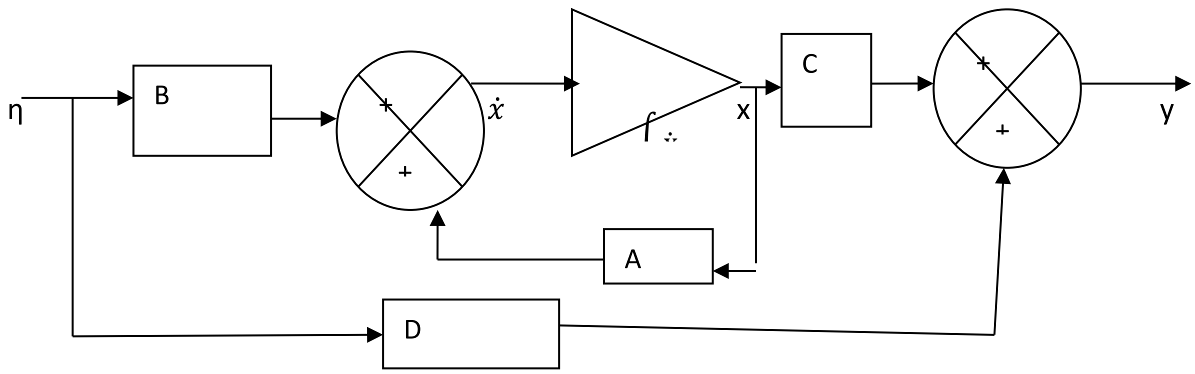


Fig 5.2.1: Block diagram representation of State equation.

The state equations are set of first order differential equations. The matrices A & B may be either constant or functions of time. For aircraft equation of motion, the matrices are composed of an array of constants. The constants making up either A or B matrices are the stability & control derivatives of the airplane. If governing equations are of higher order, they can be reduced to a system of first order differential equations. For example suppose the physical system being modeled can be described by an nth order differential equation.

$\frac{d^n}{dt^n}c(t)$ + a_1 $d^{n-1}c(t)/ dt^{n-1}$ + a_2 $d^{n-2}c(t)^{)}/dt^{n-2}$+…+a_{n-1} dc(t)/dt +a_n c(t) = r(t)

The variable c(t), r(t) are output & input variables respectively. The above differential equation can be reduced to a set of first-order differential equation by defining the state variable as follows:

x_1 (t) =c(t)

x_2 (t) = dc(t)/dt

$\vdots$

$x_n(t) = d^{n-1}$ $c(t)/dt^{n-1}$

The state equation can be written as

$\dot{x}_1$ (t) = x_2(t)

$\dot{x}_2$(t) = x_3(t)

$\vdots$

$\dot{x}_n$ (t) = $-a_n x_1(t) - a_{n-1} x_2(t) - \ldots - a_1 x_n$ (t) + r(t)

Rewriting the equation in the state vector form yields

$\dot{x}$ = A x + B η

Where A & B are as shown below:

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & & \dots & -a_1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Output equation y= C x ; Where C= $[1 \quad 0 \quad 0 \quad \dots 0 \quad 0]$

5.3 General form of time invariant linear system: General form of linear time invariant system is given by

$\dot{x}$ = A x + B η

y = C x + D η

y is the output. For linear time invariant system matrix, A, B, C & D are constant and do not change with time. x is the state variable matrix; η is control or input vector.

5.4 Matrix transfer function. State equations represent the complete internal description of a system where as the transfer function is only the input-output representations. Consequently, the transfer function can be obtained uniquely from the state equations.

$\dot{x}$= Ax + B u (1); where u is the input and x is state variable matrix.

Taking the Laplace transform of both sides considering zero initial conditions, we get

s X(s) = A x(s) + B u(s) (2)

∴ X (s) = (s I – A)$^{-1}$ B U(s) (3)

Output equation is

y = C x + D u; substituting the value of X(s) from equation (3) into Laplace transform of output equation;

Y(s) = [C (s I-A)$^{-1}$ B + D] U(s) (4)

Transfer function is obtained as

G(s) = Y(s)/U(s) = C(s I-A)$^{-1}$ B + D (5)

G(s) is called matrix transfer function.

Example: Let A = $\begin{bmatrix} -2 & 0 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$; B = $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; C = $[2 \quad 1 \quad -1]$; D =0

Determine the matrix transfer function.

Solution:

s I – A = $\begin{bmatrix} s+2 & 0 & -1 \\ -1 & s+2 & 0 \\ -1 & -1 & s+2 \end{bmatrix}$

Matrix of co-factors = $\begin{bmatrix} s2+3s+2 & s+1 & s+3 \\ 1 & s2+3s+1 & s+2 \\ s+2 & 1 & s2+4s+4 \end{bmatrix}$

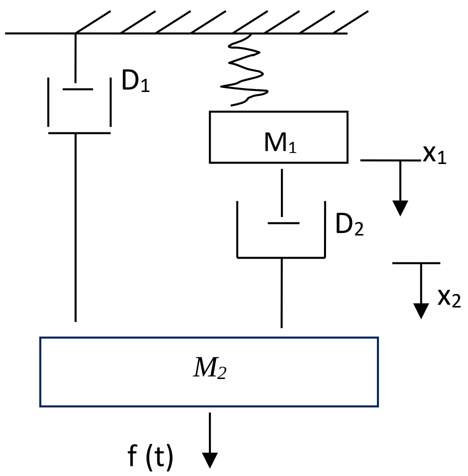
Adjoint (s I –A) = $\begin{bmatrix} s2+3s+2 & 1 & s+2 \\ s+1 & s2+3s+1 & 1 \\ s+3 & s+2 & s2+4s+4 \end{bmatrix}$

Determinant (s I – A) = (s+2) (s^2 + 3s +2) –s-3 = s^3+5s^2+7s+1

∴ G(s) = $\frac{[2 \quad 1 \quad -1]}{det(sI-A)} \begin{bmatrix} s2+3s+2 & 1 & s+2 \\ s+1 & s2+3s+1 & 1 \\ s+3 & s+2 & s2+4s+4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

= $(s^2+4s+3)/(s^3+5s^2+7s+1)$ Answer.

Example of state space modeling of dynamical system: A mechanical system with two - degree of freedom is shown in Fig 5.5. Derive the state equation of the system.



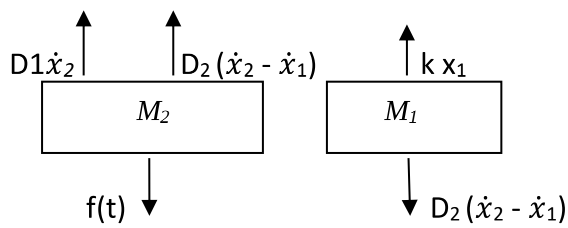


Fig 5.6 Free Body Diagram

Fig 5.5 Mass spring damper system with two- degree of freedom

Solution: Free body diagram is shown in fig 5.6.

Writing the differential equation for mass M_2

$M_2 \frac{d2y}{dt2}$ + $(D_1 + D_2)\frac{dx2}{dt}$ $-D_2 \frac{dx1}{dt}$ = f(t) (1)

$M_1 \frac{d2x1}{dt2}$ + $D_2\frac{dx1}{dt} + kx1$ $-D_2 \frac{dx2}{dt}$ = 0 (2)

We can transform them into a set of four Ist order differential equation by defining two more state variables.

$x_3 = dx_1/dt$

$\dot{x}1=x_3$ (3)

$x_4 = dx_2/dt$

$\dot{x}_2= x_4$ (4)

Substituting these into equation (1) we get

$M_2 \frac{dx4}{dt}$ + $(D_1 + D_2)\, x_4 - D_2\, x_3$ = f (t) (5)

$M_1 \frac{dx3}{dt}$ + $D_2\, x_3 + kx_1 - D_2x_4$ = 0 (6)

From equation (5) and (6) we get

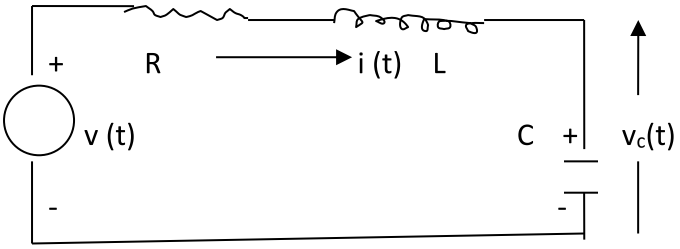
$\dot{x}_4 = -((D_1 + D_2)/ M_2)\, x_4 + D_2\, x_3 + f(t)/ M_2$ (7)

$\dot{x}_3 = -(D_2\, x_3 - kx_1 + D_2x_4)/ M_1$ (8)

Hence using equation (3), (4), (7) and (8), state equations are

$$\begin{bmatrix} \dot{x}1 \\ \dot{x}2 \\ \dot{x}3 \\ \dot{x}4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/M1 & 0 & -D2/M1 & D2/M1 \\ 0 & 0 & D2/M2 & -(D1+D2)/M2 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \\ x4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M2 \end{bmatrix} \text{f (t)}$$

Examples of State equation modeling of an Electrical Circuit: Consider an electrical network shown below. Find the state space equation, if input voltage is v (t) and output is $v_c(t)$. Resistance is R and inductance is L.



Solution: v (t) = R i(t) + L di/dt + v_c (t) (1)

Let i(t) and $v_c(t)$ be defined as state variables

x_1 = i(t)

x_2 = $v_c(t)$

i(t) = C d $v_c(t)$/dt

i.e. C d x_2 /dt = x_1

$\dot{x}_2 = x_1/C$

From equation (1) we get

v(t) = R x_1 + L $\dot{x}_1$ + x_2

Hence,

$\dot{x}_1$ = - x_1 R/L - x_2 /L + v (t)/L

$\dot{x}_2 = x_1/C$; Hence state equation is

$$\begin{bmatrix}\dot{x}1\\ \dot{x}2\end{bmatrix} = \begin{bmatrix}-R/L & -1/L\\ 1/C & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1/L\\ 0\end{bmatrix} v(t)$$

$$A = \begin{bmatrix}-R/L & -1/L\\ 1/C & 0\end{bmatrix};\ B = \begin{bmatrix}1/L\\ 0\end{bmatrix}$$

Output equation: $y = x_2$; $y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

5.5 State Transition matrix, matrix exponential-properties:

5.5.1 State transition matrix: The state transition matrix is defined as the matrix that satisfies the linear homogeneous state equation i.e.

$\dot{x}$ = Ax; Homogeneous state equation.

x (0) = $\begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix}$; Initial state at time t = 0.

x(t) = ∅ (t) x(0); where ∅ (t) is the state transition matrix.

5.5.2 State transition matrix by Laplace Transform.

$\dot{x}$ = Ax;   x (0) = = $\begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix}$

Taking Laplace transform of the above equation, we get

s x(s) –x (0) = Ax(s)

(s I-A) x(s) = x(0)

∴ x (s) = (s I-A) $^{-1}$ x(0)

The state transition matrix is obtained by taking the inverse Laplace transform of the above equation.

∅ (t) = L $^{-1}$ (s I – A) $^{-1}$

5.5.3 The state transition matrix by classical technique. State transition matrix can be found in the following manner.

x (t) = e^{At} x (0)

Where e^{At} is a matrix exponential & d (e^{At})/dt = A e^{At}. Substituting the above equation into homogeneous state equation shows that it is a solution.

Ae^{At} x (0) = A e^{At} x(0)

e^{At} can be reduced by power series as follows:

e^{At} = I + A t + A^2 t^2 /! 2 + A^3 t^3/! 3 + ⋯

∅ (t)   = e^{At} = I + A t + A^2 t^2 /! 2 + A^3 t^3/! 3 + ⋯

5.5.4 Properties of the state Transition Matrix:

1. ∅ **(0) = e^{A0} = I**

2. $[\emptyset\ (t)]^{-1} = [\emptyset\ (-t)]$

3. $\emptyset\,(t_1 + t_2) = e^{A\,(t1+t2)} = \emptyset(t_1)\ \emptyset\,(t_2)$

4. $[\emptyset(t)]^{k} = \emptyset\,(kt)$

5.6. Solutions of state equation. Once the state transition matrix has been found, the solution to the nonhomogeneous equation can be determined as follows:

$\dot{x} = A\,x + B\,\eta$

Taking the Laplace transform of both sides

$S\,x(s) – x\,(0) = A\,x(s) + B\,\eta\,(s)$

Solving for x(s)

$x\,(s) = (sI\text{-}A)^{-1}\,x(0) + (sI – A)^{-1}\,B\,\eta\,(s)$

Hence, $x(t) = \emptyset\,(t)\,x(0) + L^{-1}\,(s\,I\text{-}A)^{-1}\,B\,\eta\,(s)$

$$\mathbf{x\,(t) = \emptyset(t)\,x(0)} + \int_0^t \emptyset(t-\tau)B\,\eta\,(\tau)\,d\tau$$

5.7 Numerical Solution of State Equations. The complete solution of the state equations was shown to be

$$\mathbf{x\,(t) = \emptyset(t)\,x(0)} + \int_0^t \emptyset(t-\tau)B\,\eta\,(\tau)\,d\tau \qquad \textbf{(1)}$$

The solution of equation (1) can be obtained numerically by replacing the continuous system by discrete time system. A sampling interval Δt is specified so that

$k\,\Delta t < t < (k + 1)\,\Delta t$

The equation (1) can be rewritten as

$$x_{k+1} = e^{A\Delta t}x_k + e^{A\Delta t}\int_0^{\Delta t} e^{-A\tau}B\,\eta(\tau)\,d\tau \qquad (2)$$

If we assume the control vector $\eta(\tau)$ is constant over the time interval Δt then the integral can be evaluated

$$\int_0^{\Delta t} e^{-A\tau}B\,\eta(\tau)\,d\tau = (I - e^{-A\Delta t})\,A^{-1}\,B\,\eta_k \qquad \textbf{(3)}$$

Substituting the solution of the integral back into equation (2) yields

$$x_{k+1} = e^{A\Delta t}\,x_k + [e^{A\Delta t} - I]\,A^{-1}\,B\,\eta_k \qquad (4)$$

This equation can be simplified further by letting

$$M = e^{A\Delta t} \qquad (5)$$

N = ($e^{A\Delta t}$ - I) A $^{-1}$ B (6)

The solution vector can now be expressed as

$x_{k+1} = M x_k + N \eta_k$ (7)

Equation (7) can be used to determine the time domain solution; for example

$x_1 = Mx_0 + N \eta_0$

$x_2 = M x_1 + N \eta_1$

$x_3 = Mx_2 + N \eta_2$

.

.

.

$x_{k+1} = M x_k + \eta_k$

On combining these equations, one obtains

$x_k = M_k x_0 + \sum_{i=0}^{k-1} M^{k-1-i} N \eta_i$

Once a satisfactory time interval is selected the matrices M and N need be calculated only one time. These matrices can be evaluated by the matrix expression

$M = e^{A\Delta t} = I + A\Delta t + \frac{1}{!2} A^2 \Delta t^2 \ldots$

$N = \Delta t \,(I + + \frac{1}{!2} A \Delta t + \frac{1}{!3} A^2 \Delta t^2 \ldots) B$

The number of terms required in the series expansion depends on the time interval Δt .

5.8 Canonical transformation of state equations-significance-Eigen values-real distinct, repeated, complex.

5.8.1 Canonical transformation (Diagonal Matrix) of state equations- significance: In formulating a physical system into space-space representation we must select a set of state variables to describe the system. The set of state variable we select may not be the most convenient from the point of the mathematical operations we need to perform to determine the solution of state equations. It is possible to define a transformation matrix, P, which will transform the original state equations into a more convenient form. To examine the characteristics of a given state equation it is useful to have the state equations in a canonical form where the plant matrix is diagonal matrix. In canonical form the state equations are decoupled. Further state transition matrix can be easily found once the state equations are transformed into canonical form.

5.8.2 Method of Canonical Transformation. Consider a system that can be modeled by this state equation:

$\dot{x}$ = A x + B η (1)

y= C x (2)

Where the plant matrix A is not diagonal matrix. Defining a new state vector z so that x and z are related by way of a transformation matrix P,

x = P z (3)

Rewriting the state equation in terms of the new state vector z yields

$\dot{z} = P^{-1} A P z + P^{-1} B \eta$ (4)

This can be written as

$\dot{z} = \Lambda z + \bar{B} \eta$ (5)

$y = \bar{C} z$ (6)

Where Λ is a diagonal matrix. The matrices Λ, $\bar{B}$, and $\bar{C}$ are defined as

$\Lambda = P^{-1} A P$ (7)

$\bar{B} = P^{-1} B$ (8)

$\bar{C} = CP$ (9)

The transformed state equation has the same form as the original equation. If the transformation matrix P is chosen such that Λ is a diagonalized matrix then the equations is in canonical form.

The transformation matrix P is determined from the eigenvectors of the plant matrix A. As has been shown earlier the Eigen values of A are determined by solving the following **characteristic equations**:

$|\boldsymbol{\lambda I - A}| = \mathbf{0}$ (10)

This yields the **characteristic equation**

$\lambda^n + an\, \lambda^{n-1} + a_{n-1} \lambda^{n-2} + \ldots + a_2 \lambda + a_1 = 0$ (11)

The roots of the characteristic equation are the eigenvalues of the system. The eigenvectors can be determined by solving the equations

$(\lambda_i I - A)P_i = 0$ where i = 1,2,3,…,n (12)

The transformation matrix P is formed from the eigenvectors of the plant matrix. The eigenvectors form the columns of the transformation matrix as

$P = [P_1\ P_2\ P_3 \ldots P_n]$ (13)

5.8.2.1 Real Distinct Eigenvalues. For these non- repeated real eigenvalues, the transformation matrix P depends on the eigenvalues of the plant matrix A. If the eigenvalues of A are real and distinct, the transformation matrix P is made up of the eigenvectors of A as follows:

$P = [P_1\ P_2\ P_3 \ldots P_n]$

We illustrate how the transformation is determined by the following example problem

Example problem: Given the following state equations, determine the transformation matrix P so that new state equations are in the state canonical form.

$$\begin{bmatrix} \dot{x}1 \\ \dot{x}2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} [u]$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution: First find the eigenvalues of A:

$|\lambda I - A| = 0$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{vmatrix} = 0$$

Or $\lambda^2 + 3\lambda + 2 = 0$

$\therefore \lambda = -2$ and $\lambda = -1$

The eigenvector for $\lambda = -1$ is found using equation (12):

$(\lambda_i I - A)P_i = 0$

$\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right)\begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix} = 0$

$-P_{11} - P_{21} = 0$

$2 P_{11} + 2P_{21} = 0$

Both equations yield the same relationship between P_{11} and P_{21} . we will arbitrarily select

$P_{11} = 1$

$P_{21} = -1$

The eigenvector for λ = -1 is

$P_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

In similar manner we can obtain the eigenvector for λ = -2. Solving equation (12) yields the following equation

$\left(\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right)\begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix} = 0$

Or $-2P_{12} - P_{22} = 0$

$2P_{12} + 1P_{22} = 0$

Again, we will specify $P_{12} = 1$ and then solve for P_{22}. The eigenvector P_2 becomes

$P_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

The transformation matrix P now can be constructed by stacking the eigenvectors as follows:

$P = [P_1 \; P_2]$

$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

To determine the new state equation we need the inverse of P:

$P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

The diagonal matrix Λ is defined in terms of P and A:

$\Lambda = P^{-1} A P$

$\Lambda = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

Where the eigenvalues are on the diagonal.

In a similar manner $\bar{B}$ and $\bar{C}$ can be found.

$\bar{B} = P^{-1} B$

$\bar{B} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$\bar{B} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$\bar{C} = CP$

$\bar{C} = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

$\bar{C} = \begin{bmatrix} 2 & 1 \end{bmatrix}$

New state equations are:

$\begin{bmatrix} \dot{z}1 \\ \dot{z}2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} [u]$

$y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$\begin{bmatrix} z_1\,(0) \\ z_2\,(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

This example demonstrates an important property of canonical transformation. The eigenvalues and corresponding characteristic equation remain unchanged. The transformed plant matrix is purely diagonal matrix having the eigenvalues of the original A matrix along the diagonal. The state transition matrix can be shown to be the following:

$\emptyset\,(t) = e^{\Lambda t} = \begin{bmatrix} e^{\,t\lambda_1} & 0 \\ 0 & e^{\,t\lambda_2} \end{bmatrix}$ or

$\emptyset\,(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2\,t_2} \end{bmatrix}$

The solution of the transformed state equation would be:

z (t) = $\emptyset$(t) z(0) + $\int_0^t \emptyset(t-\tau)\bar{B}\,\eta\,(\tau)$ dτ

$$\begin{bmatrix} z_1\,(t) \\ z_2\,(t) \end{bmatrix} == \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2\ t_2} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} \int_0^t 2e^{-(t-\tau)}\ d\tau \\ -\int_0^t 2\,e^{-2(t-\tau)}\ d\tau \end{bmatrix} = \begin{bmatrix} 2-e^{-t} \\ -1 \end{bmatrix}$$

The output of the system is given by

y= $\bar{C}$z = $[2 \quad 1]\begin{bmatrix} 2-e^{-t} \\ -1 \end{bmatrix}$

y= 3-2e^{-t}

5.8.2.2 Repeated Eigenvalues: Where the eigenvalues are repeated, the procedure outlined for the distinct eigenvalues produces a singular transformation matrix. The eigenvectors for the repeated roots are the same; therefore, two or more columns of the transformation matrix are identical, which results in a nonsingular matrix. For repeated eigenvalues an almost diagonal matrix, called a Jordan matrix, can be obtained. The Jordan matrix is.

$$\Lambda = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

Notice that the diagonal immediately above the repeated eigenvalues is composed of ones. The eigenvectors associated with the distinct eigenvalues are determined as before. For the repeated eigenvalues the eigenvectors are determined using the following relationships:

$(\lambda_i\, I - A)P_1 = 0$

$(\lambda_i\, I - A)P_2 = -\,P_1$ (14)

$(\lambda_i\, I - A)P_m = -P_{m-1}$

Example Problem: Given the state-space equations

$\dot{x}$ = A x + B η

Where

$$A = \begin{bmatrix} 0 & -1 & -3 \\ -6 & 0 & -2 \\ 5 & -2 & -4 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Determine the transformation matrix P so that the new state equations are in the Jordan canonical form.

Solution: The transformation matrix P is determined from the eigenvectors of the A matrix:

$|\lambda I - A|= 0$

$$\begin{vmatrix} \lambda & 1 & 3 \\ 6 & \lambda & 2 \\ -5 & 2 & \lambda+1 \end{vmatrix} = \lambda^3 + 4\lambda^2+5\lambda +2$$

The roots of the characteristic equation are λ = -2, λ = -1, λ = -1. We have a repeated eigenvalue λ = -1. The eigenvalues for the repeated roots are determined using equation (14):

$(\lambda_i I – A)P_1 = 0$

$(\lambda_i I – A)P_2 = - P_1$

The eigenvector P1 is determined from the following equations

$$\begin{bmatrix} -1 & 1 & 3 \\ 6 & -1 & 2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{21} \\ P_{31} \end{bmatrix} = 0$$

$-P_{11} + P_{21}+ 3 P_{31} = 0$

$6P_{11}-P_{21} + 2P_{31}=0$

$-5P_{11} + 2 P_{21} + 3 P_{31} = 0$

From the first two equations we can eliminate P_{21}:

$5P_{11} + 5P_{32} = 0$

Let $P_{11} = 1$ then $P_{31} = -1$

From the first equation

$-P_{11} + P_{21}+ 3 P_{31} = 0$; or

$P_{21} = P_{11} -3P_{31} = 4$

The eigenvector P_1 is as follows:

$$P_1 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$$

The second eigenvector for λ = -1 is determined from the equation $(\lambda_i I – A) P_2 = - P_1$:

$-P_{12} + P_{22} + 3P_{32} = -1$

$6P_{12} - P_{22} + 2P_{32} = -4$

$-5P_{12} + 2P_{22} + 3P_{32} = 1$

Eliminating P_{22} from the two equations yields

$5P_{12} + 5P_{32} = -5$

Let $P_{12} = 1$, therefore $P_{32} = -2$. Substituting P_{12} and P_{32} into the first equation yields P_{22}:

$P_{22} = -1 + P_{12} - 3P_{32} = 6$

The second eigenvector is

$$P_2 = \begin{bmatrix} 1 \\ 6 \\ -2 \end{bmatrix}$$

The eigenvector for the distinct eigenvalue $\lambda = -2$ is found in usual way:

$$P3 = = \begin{bmatrix} 1 \\ 2.75 \\ -0.25 \end{bmatrix}$$

The transformation matrix P is formed by stacking the eigenvector:

$P = [P_1\ P_2\ P_3]$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 6 & 2.75 \\ -1 & -2 & -0.25 \end{bmatrix}$$

5.9 Controllability and Observability-definition-significance:

5.9.1 Controllability- Definition & Significance: Controllability is concerned with whether the states of the dynamic system are affected by the control input. A system is said to be completely controllable if there exists a control that transfers any initial state x_i (t) to any final state x_f (t) in some finite time. If one or more of the states are unaffected by the control, the system is not completely controllable. Controllability plays important role in design of control system. If a system is state controllable, then it is possible to use a linear control law to achieve a specific eigenvalue.

A mathematical definition of controllability for a linear dynamic system can be expressed as follows:

If the dynamic system can be described by the state equation:

$\dot{x}$ **= A x + B η** where x and η are the state and control vectors of the order n and m, respectively, then the necessary and sufficient condition for the system to be completely controllable is that the rank of the matrix P is equal to the number of states. The matrix P is constructed from the A & B matrices in the following ways:

P = [B, AB, A^2B, … A^{n-1} B]

The rank of a matrix is defined as the largest non-zero determinant.

5.9.2 Observability-Definition & Significance: Observability deals with whether the state of the system can be identified from the output of a system. A system is said to be completely observable if any state x can be determined by the measurement of the output y (t) over a finite time interval. If one or more states cannot be identified from the output of the system, the system is not observable. Observability plays an important role in design of **state observer** which is used when it is not possible to measure a particular state due to various reasons.

A mathematical test for the observability of an n^{th} order system given by the equations:

$\dot{x}$ **= A x + B η**

y = Cx + D η

is given as follows:

The necessary and sufficient condition for a system to be completely observable is that the matrix U, defined as

U = [C^T, A^T C^T… ($A^{T)}$ $^{n-1}$ C^T] is of the rank n.

Example problem 1: Determine whether the system that follows is state controllable and observable. The A, B and C matrices of the state and output equation are

A =$\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$

B = $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

C =$\begin{bmatrix} 1 & 0 \end{bmatrix}$

Solution: The controllability matrix, V, is defined for this problem as:

V = $[B \quad AB]$

AB = $\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ = $[-5]$

V = $\begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix}$

The rank of V is of the same as the order of the system. Therefore the system is state controllable.

The observability matrix, U, for this example is

U = [C^T, A^T C^T]

A T C T = $\begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ = $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

U =$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The rank of the observability matrix also is of the same order of the system. Therefore the system is state observable.

Example problem 2: Consider the system represented by the following equation

$\begin{bmatrix} \dot{x}1 \\ \dot{x}2 \end{bmatrix}$ =$\begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ + $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ [u]

Determine whether the system is state controllable.

Solution: For a second-order system the controllability matrix is defined as

V= [B AB] ; The matrix product AB follows:

AB = $\begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$= $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$

The controllability matrix can now be expressed as ; V = $\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$

The determinant of V is 0, which means the rank of the matrix is less than the order of the system. Therefore, the system is not state controllable.

5.10 Digital Control-Overview, Advantages and Disadvantages:

5.10.1 Digital Control Overview and Implementation: A digital control takes an analog signal, samples it with an analog to digital converter (A/D), processes the information in the digital domain, and the converts the signal to analog with a digital-to-analog converter. The key here is to provide redundant paths in the event of hard ware failure. An overall digital flight control block diagram is shown below in Fig 5.10. Here the signal comes from a sensing device, such as gyro. Next, it is fed in parallel along multiple paths to an analog to digital (A/D) converter. After the signal is in the digital form, the flight control computer executes the control algorithms. The output from the flight control computers is then fed to a digital-to-analog (D/A) converter, which in turn operate an actuator.

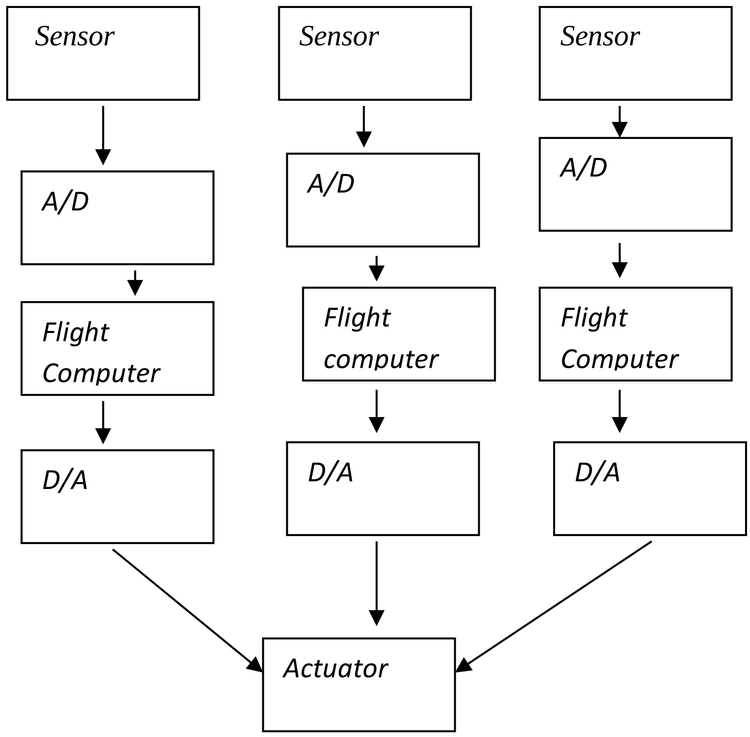


Fig 5.10: Block diagram of a Digital Control Implementation

5.10.2 Digital Control Advantages:

1. They are more versatile than analog because they can be easily programmed without changing the hardware.

2. It is easy to implement gain scheduling to vary flight control gains as the aircraft dynamics change with flight conditions.

3. Digital components in the form of electronic parts, transducers and encoders are often more reliable, more rugged, and more compact than analog equipment.

4. Multi-mode and more complex digital control laws can be implemented because of fast, light, and economical micro-processors.

5. It is possible to design "Robust" controller that can control the aircraft for various flight conditions including some mechanical failures.

6. Improved sensitivity with sensitive control elements that require relatively low energy levels.

5.10.3 Disadvantages of Digital Control.

1. The lag associated with sampling process reduces the system stability.

2. The mathematical analysis and system design of a sampled data system is more complex.

3. The signal information may be lost because it must be digitally reconstructed from an analog signal.

4. The complexity of the control process is in the software implemented control algorithm that may contain error.

5. Software verification becomes critical because of the safety of flight issue. Software errors can cause the aircraft to crash.

Note: For additional numerical problems and solutions on unit V and unit II/I see Appendix 'A'.

Appendix 'A'. Numerical Problems

Numerical Problems on unit V:

Problem 1: Obtain the state transition matrix φ (t) of the following system

$\begin{bmatrix}\dot{x}1\\ \dot{x}2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$; Obtain the inverse of the state transition matrix $\varphi^{-1}(t)$.

Solution: For this system A = $\begin{bmatrix}0 & 2\\ -2 & -3\end{bmatrix}$

sI –A = $\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix} = \begin{bmatrix}s & -1\\ 2 & s+3\end{bmatrix}$

$(sI - A)^{-1} = \frac{1}{(s+1)(s+2)}\begin{bmatrix}s+3 & 1\\ -2 & s\end{bmatrix} = \begin{bmatrix}\frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)}\\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)}\end{bmatrix}$

$\varphi(t) = e^{At} = L^{-1}(sI-A)^{-1}$;

$= \begin{bmatrix}2e^{-t} - e^{-2t} & e^{-t} - e^{-2t}\\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t}\end{bmatrix}$

Now $\varphi^{-1}(t) = \varphi(-t) = \begin{bmatrix}2e^{t} - e^{2t} & e^{t} - e^{2t}\\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t}\end{bmatrix}$ (Replace t with –t in the matrix of $\varphi(t)$). Answer.

Problem 2: Obtain the time response of the following:

$\begin{bmatrix}\dot{x}1\\ \dot{x}2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$; u is a unit step function.

Solution: For this system A = $\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$; B = $\begin{bmatrix}0\\ 1\end{bmatrix}$

$x(t) = L^{-1}(sI-A)^{-1}x(0) + L^{-1}(sI-A)^{-1} B\, U(s)$

$L^{-1}(sI-A)^{-1}$ was found in the last example problem 1.

$= \begin{bmatrix}2e^{-t} - e^{-2t} & e^{-t} - e^{-2t}\\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t}\end{bmatrix}$

Now let us find out L^{-1} (s I –A) -1 B U(s) $= \begin{bmatrix}\frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)}\\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} 1/s = \begin{bmatrix}\frac{1}{s(s+1)(s+2)}\\ \frac{1}{(s+1)(s+2)}\end{bmatrix}$

$= \begin{bmatrix}\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\\ \frac{1}{s+1} - \frac{1}{(s+2)}\end{bmatrix}$

$\therefore L^{-1}(sI-A)^{-1} B\, U(s) = \begin{bmatrix}\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\\ e^{-t} - e^{-2t}\end{bmatrix}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

If x(0) = 0 then x(t) can be simplified to

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$ **Answer**.

Problem 3: Obtain the response y (t) of the following system.

$$\begin{bmatrix} \dot{x}1 \\ \dot{x}2 \end{bmatrix} = \begin{bmatrix} -1.0 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$ u; u is a unit step function.

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$; $y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Solution: simplifying the state equation we get;

$\dot{x}_1 = -1x_1 - 0.5\, x_2 + 0.5\, u$ (1)

$\dot{x}_2 = x_1$ (2)

$y = x_1$ (3)

Taking Laplace transform of equation (1) & (2)

$sx_1(s) = -x_1(s) - 0.5\, x_2(s) + 0.5\, u(s)$ (4)

$sx_2(s) = x1(s)$; $\therefore\ x_2(s) = x_1(s)/s$ (5)

Substituting the value of x2 (s) from equation (5) into equation (4) we get

$s\, x_1(s) = -x_1(s) - 0.5\, x_1(s)/s + 0.5/s$

$\therefore (s+1)\, x_1(s) + x_1(s)/2s = 1/2s$

$\therefore x_1(s) = 1/(2s2 + 2s + 1) = 1/(2(s^2 + s + ½)) = 1/(2(s + ½)^2 + (1/2)^2)$

$\therefore x_1(t) = e^{-0.5t}$ Sin (0.5t)

$\therefore y = x_1(t) = e^{-0.5t}$ Sin (0.5t) **Answer**

Problem 4: Consider the system given by

$\dot{\boldsymbol{x}}$ **= A x + B u**

y = C x

where $A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$; Obtain the transfer function Y(s)/U(s).

Solution: We know that transfer function is given by

Y(s)/U(s) = C (s I-A)$^{-1}$ B + D ; in this problem D=0

Hence TF = C (s I-A)$^{-1}$ B

sI-A = = $\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix}$ - $\begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ = $\begin{bmatrix} s+1 & 0 & -1 \\ -1 & s+2 & 0 \\ 0 & 0 & s+3 \end{bmatrix}$

$|sI - A|$= (s+1)(s^2+5s+6)

Matrix of co-factors of (sI-A)

= $\begin{bmatrix} (s+2)(s+3) & s+3 & 0 \\ 0 & (s+1)(s+3) & 0 \\ s+2 & 1 & (s+1)(s+2) \end{bmatrix}$

Adj (sI-A) = $\begin{bmatrix} (s+2)(s+3) & 0 & s+2 \\ s+3 & (s+1)(s+3) & 1 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}$

Adj (sI-A) B =$\begin{bmatrix} (s+2)(s+3) & 0 & s+2 \\ s+3 & (s+1)(s+3) & 1 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

=$\begin{bmatrix} s+2 \\ 1 \\ (s+1)(s+2) \end{bmatrix}$

C. Adj(sI-A). B $\quad$ $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} s+2 \\ 1 \\ (s+1)(s+2) \end{bmatrix}$ =s + 2+1 = s+3

$\therefore \frac{C\ Adj\ (sI-A)B}{|sI-A|} = \frac{s+3}{(s+1)(s+2)(s+3) =} \frac{1}{(s+1)(s+2)}$

$\therefore$ Y(s)/U(s) = $\frac{1}{(s+1)(s+2)}$ **Answer**.

Problem 5: Given the following state equation

$\begin{bmatrix} \dot{x}1 \\ \dot{x}2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$u ; .

Y= $\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .Determine the system response if u is a unit step function.

Solution: From the given state equation matrix, we get

$\dot{x}_1$ = x2 (1)

$\dot{x}_2$= -2x_1 -3x_2 + 2u (2)

Taking the Laplace transform of equation (1) & (2) we get

s $x_1(s) - x_1(0) = x_2(s)$ (3)

s $x_2(s) - x_2(0) = -2x_1(s) - 3x_2(s) + 2U(s)$ (4)

Now $x_1(0) = 0$ and $x_2(0) = 1$; substituting these values in equation (3) & (4) we get

s $x_1(s) = x_2(s)$ (5)

$\therefore x_1(s) = x_2(s)/s$ (6)

Substituting the value of $x_2(0)$, U(s) and $x_1(s)$, into equation (4) we get

s $x_2(s) - 1 = -2x_2(s)/s - 3x_2(s) + 2/s$; which after simplification yields

$x_2(s) = s+2/(s^2 + 3s + 2) = \frac{1}{s+1}$

$\therefore x_2(t) = e^{-t}$

Now using equation (5) we get

s $x_1(s) = x_2(s)$; or $x_1(s) = x_2(s)/s = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$

Taking Laplace inverse we get

$x_1(t) = 1 - e^{-t}$

$\therefore x(t) = \begin{bmatrix} 1 - e^{-t} \\ e^{-t} \end{bmatrix}$

& $y = [3 \quad 1] \begin{bmatrix} 1 - e^{-t} \\ e^{-t} \end{bmatrix} = 3 - 3e^{-t} + e^{-t} = 3 - 2e^{-t}$ **Answer**.

Problem 6: A control system is described by the differential equations

$d^3 y(t)/dt^3 = u(t)$ where y(t) is output, u(t) = input.

Describe the system in $\dot{x}$ **= A x + B u ; y = C x + Du** form. Calculate the state transition matrix e^{At} **of the system.**

Solution:

$d^3 y(t)/dt^3 = u(t)$

Let $x_1(t) = y(t)$; $x_2(t) = dy(t)/dt$; $x_3(t) = d^2y(t)/dt^2$;

$\dot{x}_1 = x_2$;

$\dot{x}_2 = x_3$;

$\dot{x}_3 = u(t)$

$$\therefore \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t); \quad \dot{\boldsymbol{x}} = \mathbf{A\ x + B\ u;}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u; \quad \mathbf{y = C\ x + \ Du}$$

State transition matrix is calculated as follows:

$$\Phi = \boldsymbol{e^{At}} = \mathbf{I + A\ t + A^2\ t^2 /!\ 2 + A^3\ t^3/!\ 3} + \cdots$$

$$At = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} t = \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}$$

$$(At)^2 = \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(At)^3 = (At)^2 \ At = \begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \Phi = \boldsymbol{e^{At}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{t^2}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{Answer}.$$

Problem 7: (a) For the given TF construct state space model

$$TF = y/u = \frac{b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

(b) Construct state space model for the system

$$\ddot{y} + 5\,\dot{y} + 6y = u$$

Solution:

(a) Above equation can be written as

$$y(s^3 + a_2 s^2 + a_1 s + a_{0)} = b_0\ u$$

Taking the inverse Laplace transform of both sides, we get

$d^3 y/dt^3 + a_2 d^2y/dt^2 + a_1\, dy/dt + a_0 y = b_0\, u$

Let $x_1 = y$

$x_2 = dy/dt$

$x_3 = d2y/dt^{2}$;

Hence

$\dot{x_1} = x_2$;

$\dot{x_2} = x_3$;

$\dot{x_3} = -a_2 x_3 - a_1 x_2 - a_0 x1 + b_0 u$

Hence ∴ $\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_0 \end{bmatrix} u$

$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(b) $\ddot{y} + 5\,\dot{y} + 6y = u$

Let $x_1 = y$; $x_2 = dy/dt$

Hence $\dot{x_1} = x_2$; $\dot{x_2} = -5x_2 - 6x_1 + u$;

$\begin{bmatrix} \dot{x}1 \\ \dot{x}2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ **Answer.**

Problem 8: For the differential equation that follows, rewrite the equation in state space formulation.

(a) $d^2 c(t)/dt^2 + 2\xi\omega_n\, dc(t)/dt + \omega_n^2 = r$
(b) $d^3 c/ dt^3 + d^2c/dt^2 + 2\, d\, c/dt + c = 2\, dr/dt + 3r$
(c) $d^{2}\theta/dt^2 + 3\, d\theta/dt + 2\, d\alpha/dt + 5\alpha = -6\delta e$.

$d\alpha/dt + 4\alpha - 15\, d\theta/dt = -3\delta e$

Solution:

(a) Let $x_1 = c$
$x_2 = dc/dt$

$\dot{x_1} = x_2$;

$\dot{x_2} = -2\xi\omega_n x_2 - \omega_n^2 x_1 + r$

Hence $\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ {\omega_n}^2 & -2\xi\omega n\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}$ r **Answer**.

(b) Taking Laplace transform of both side of differential equation, we get

(s^3 + s^2 +2s +1) C(s) = (2s + 3) R(s)

C(s)/R(s) = $\frac{2s+3}{s^3+s^2+2s+1}$

$\frac{C(s)}{R(s)} = \frac{(2s^{-2}+3s^{-3})\,X(s)}{(1+s^{-1}+2s^{-2}+s^{-3})\,X(s)}$

C(s) = $(2s^{-2} + 3s^{-3})\,X(s)$

R(s) = $(1 + s^{-1} + 2s^{-2} + s^{-3})\,X(s)$

Hence, X(s) = R(s) – $(s^{-1} + 2s^{-2} + s^{-3})\,X(s)$

Let $s^{-3}\,X(s)$ = x_1 (s)

$s^{-2}\,X(s)$= x_2 (s)

$s^{-1}\,X(s)$= x_3 (s)

x_1 (s)/ x_2 (s) = 1/s

Hence, $x_2 = \dot{x_1}$; x_2 (s)/ x_3 (s) = 1/s

Hence $x_3 = \dot{x_2}$

X(s) = s x_3(s)

sx_3 (s) = R(s)- $(s^{-1} + 2s^{-2} + s^{-3})\,X(s)$ = R(s) –(x_3 (s) + 2 x_2(s) + x_1 (s))

Hence, $\dot{x_3}$ = r – (x_3 + 2 x_2 + x_1)

Therefore

$\dot{x_1} = x_2$;

$\dot{x_2} = x_3$

$\dot{x_3} = -(x_3 + 2x_2 + x_1) + u$

$y = 3x_1 + 2x_2$

Hence state space equation is

$$\therefore \begin{bmatrix}\dot{x_1}\\ \dot{x_2}\\ \dot{x_3}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} r$$

$y = [3 \quad 2 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ **Answer**.

(c)

x1 =θ

x2 = dθ/dt

x3 = α

$\dot{x_1} = x_2$

$\dot{x_2} = -3x_2 - 2\dot{x_3} - 5x_3 - 6\delta e$ (1)

$\dot{x_3} = -4x3 + 15x2 - 3\delta e$ (2)

Substituting $\dot{x_3}$ into equation (1) yields

$\dot{x_2} = -3x_2 + 8x_3 - 30x_2 + 6\delta e - 5x_3 - 6\delta e$

$\dot{x_3} = -15x_2 - 4x_3 - 3\delta e$

Hence state space equation can be written as

$$\therefore \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -33 & 3 \\ 0 & -15 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}\delta e$$

$\begin{bmatrix} y_1 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $y_1 = \theta$ & $y_3 = \alpha$ **Answer.**

Problem 9: $d^2c(t)/dt^2 + 3\,dc(t)/dt + 2\,c(t) = r(t)$

(a) Find the state transition matrix.
(b) Find the response if c(0) =1 & d c(0)/dt = 0

Solution: (a):

Let $x_1(t) = c(t)$
$x_2(t) = dc(t)/dt$
$\dot{x_1} = x_2$

$\dot{x_2} = -3x_2 - 2x_1 + r(t)$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)\,;$$

$x_1(0) = 1$; $x_2(0) = 0$

$$(sI-A) = = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$|sI - A| = s^2 + 3s + 2 = (s+1)(s+2)$

Matrix of cofactor = $\begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}$

Adj (sI-A) = $\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$

$(sI-A)^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$

$\Phi(t) = L^{-1}(sI-A)^{-1} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$ **Answer.**

$\dot{x} = A\,x + B\,u$

(sI-A) X(s) = x(0) + B U(s)

Hence $x(t) = L^{-1}(sI-A)^{-1}x(0) + L^{-1}(sI-A)^{-1}B\,U(s)$

Now $L^{-1}(sI-A)^{-1}x(0) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix}$

Now $(sI-A)^{-1}\,B\,U(s) = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ 1/s $= \begin{bmatrix} \frac{1}{(s+1)(s+2)} \\ \frac{s}{(s+1)(s+2)} \end{bmatrix}$ 1/s= $\begin{bmatrix} \frac{1}{s(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)} \\ \frac{1}{s+1} - \frac{1}{s+2} \end{bmatrix}$

Hence, $L^{-1}(sI-A)^{-1}B\,U(s) = \begin{bmatrix} \frac{1}{2} - e^{-t} + 1/2e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$

Hence, $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t} \\ -e^{-t} + e^{-2t} \end{bmatrix}$

$y = c(t) = x_1(t) = \frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t}$ **Answer.**

Numerical Problems on Unit II:

Problem 1:

(a) A transfer function is expressed as $\frac{s+16}{s^2 + 16s + 63}$. Find the poles and zeros of the transfer function. Is system stable?

(b) The transfer function of the forward path of a control system is given by $\frac{s}{s^2 + 5s + 7}$ and the transfer function of the feedback path is $\frac{10}{s+3}$. Find the closed loop transfer function of the system.

(c) Transfer function of a control system is $\frac{s}{(s+1)(s+2)}$. Find the response of the system for the unit step input.

Solution: (a)

Poles are given by the roots of characteristic equation which is obtained by equating the denominator of the transfer function with 0.

Hence, $s^2 + 16s + 63 = 0$ or (s+9) (s+7) = 0

Hence roots are s= -9 and s = -7. Therefore poles are -9 and -7. **Answer**.

Zeros are obtained by equating the numerator of the TF equal to 0.

Hence s= -16 is a zero. Hence zero is = -16 **Answer.**

Since the poles are located on the left side of s-plane, system is stable**.** **Answer**.

(b)

We know that closed loop transfer function of a control system is given by

TF of closed loop system = $\frac{G(s)}{1+G(s)(H(s)}$ (1)

Where G(s) is the transfer function in the forward path & H(s) is the TF of feedback path.

Given G(s) = $\frac{s}{s^2 + 5s + 7}$ & H(s) = $\frac{10}{s+3}$; Substituting these values in equation (1) we get

TF = $\frac{\frac{s}{s^2 + 5s + 7}}{1+\left(\frac{s}{s^2 + 5s + 7}\right)\left(\frac{10}{s+3}\right)}$ $= \frac{s^2 + 3s}{s^3 + 8s^2 + 32s + 21}$ **Answer.**

(c) TF = $\frac{s}{(s+1)(s+2)}$, Input is unit step function, hence Laplace transform of input is 1/s. Hence Laplace transform of output is y(s) = TF * Laplace transform of input.

Hence y(s) = $\frac{s}{(s+1)(s+2)}$ (1/s) = $\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$

Hence by taking the Laplace inverse of both sides we get y (t) $= e^{-t}$ -e^{-2t} **Answer**.

Problem 2: A transfer function of an aircraft is given by $\frac{p}{\delta} = \frac{6}{s+6}$ where p is roll rate and δ is aileron deflection. Analyze the behavior of the aircraft.

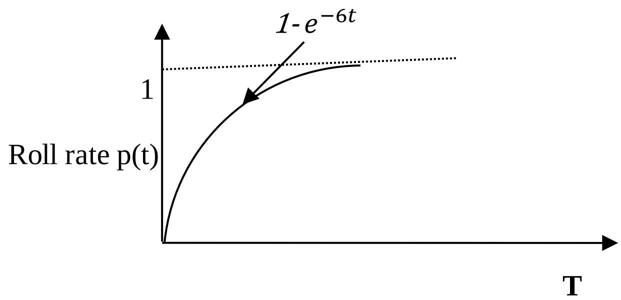
Solution: Transfer function is given by $\frac{p}{\delta} = \frac{6}{s+6}$; hence p = δ $\frac{6}{s+6}$,

.

Let δ be unit step input. So Laplace of δ(t) is = 1/s. Hence response p(s) = $\frac{6}{s(s+6)}$

= $\frac{1}{s}$ -$\frac{1}{s+6}$; Taking the inverse Laplace transform yields

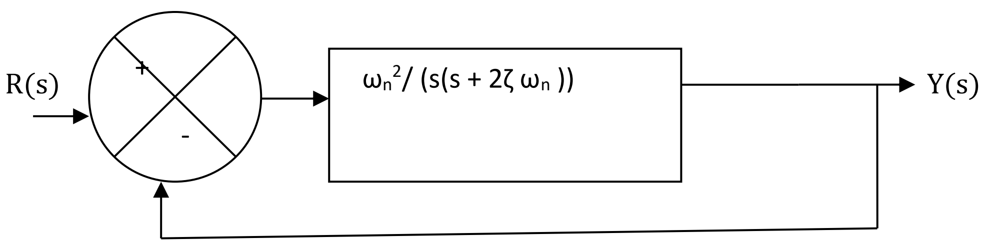
p (t) = 1-e^{-6t}; Hence for a step input of aileron, roll rate will be exponentially increasing with time constant of 1/6 seconds. Plot of roll rate with respect to time is shown below:



At time t= 0, roll rate is 0**.** It increases exponentially at t = 16 second(time constant) it reaches about 63% of the final value and after 2T it reaches 83% of the final value and after 0.5 seconds it reaches 95% of the final roll rate.

Problem 3: The characteristic equation of a closed loop system is s^2 + 25s + 300 K = 0. The reference input is a unit step function. Deduce the output for K = 5 and 10, show them in graphical form and explain.

Solution: A second order system with unity feedback is represented by the following block diagram.



G(s) = $\omega_n^2/(s(s + 2\zeta\,\omega_n))$

H(s) = 1

The closed loop transfer function is

$\frac{Y(s)}{R(s)}$ = = $\omega_n{}^2/(s^2 + 2\zeta\,\omega_n\, s + \omega n^2)$

The characteristic equation is given by

$(s^2 + 2\zeta\,\omega_n\, s + \omega n^2) = 0$ (1)

Given characteristic equation is

s^2 + 25s + 300 K = 0.

Case 1: For K = 5

We get s^2 + 25s + 1500 = 0 (2)

Comparing equation (1) and (2) yields

$\omega_n = \sqrt{1500} = 38.73$

$2\zeta\,\omega_n = 25$; $\zeta = 12.5/38.73 = 0.32$

$cos^{-1}(\zeta)$ = 71.7° ;For a step input, output is given by

Y(s) = $\omega_n{}^2/(s(s^2 + 2\zeta\,\omega_n\, s + \omega n^2))$

Taking the Laplace inverse of both sides we get

y(t) = $=1\frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega n\, t}$ $Sin\,(\omega_d\, t + cos^{-1}(\zeta))$

$where\ \omega_d = \omega_n\sqrt{1-\zeta^2}$.

Substituting the values of ζ and ω_n yields

y(t) = 1- 1.052 $e^{-12.5t}$ Sin (36.7t+71.2)

% maximum overshoot = $100\,(e^{-(\zeta/\sqrt{1-\zeta^2})\pi})$ =34.75%

Delay time $\cong \frac{1+0.7\zeta}{\omega_n}$ = 0.031 seconds

Case 2: For K = 10

Characteristic equation is

s^2 + 25s + 3000 = 0

$\omega_n = \sqrt{3000} = 54.77$

$\zeta = 12.5/54.77 = 0.228$

Output y(t) = 1- 1.027 $e^{-12.5t}$ Sin (53.32t+76.8)

%Maximum overshoot = 100 $(e^{-(\zeta/\sqrt{1-\zeta^2})\pi})$ = 100 $e^{-0.735}$ = 47.95%

Delay time is time $\cong \frac{1+0.7\zeta}{\omega_n} = 0.0211$ seconds.

Explanation:

1. System is stable for K =5 and K=10

2. Damping is less than 1 in both the cases (under damped)

3. As gain K is increased damping is reduced

4. As damping is reduced maximum overshoot increases and system becomes more oscillatory.

5. As gain is increased delay time reduces.

6. As gain is increased, un- damped natural frequency increases.

The plot of response for K = 10 is shown in fig below.